GENERALISED SPATIAL SMOOTHING USING PARTIAL ARRAYS FOR MULTIPLE TARGET BEARING ESTIMATION

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ABSTRACT

This paper considers the problem of bearing estimation for a small number of radar targets which cannot be resolved in range or Doppler frequency. Bearing estimation for non-fluctuating targets involves a single "snapshot" resulting from a multi-channel optimum (matched) filtering process. The standard spatial smoothing technique may be applied to this single-snapshot model, but only for uniform linear antenna arrays. Here we introduce a special class of nonuniform geometry with embedded *partial arrays* and a corresponding *generalised spatial smoothing* (GSS) algorithm. The partial array characteristics determine the resulting bearing estimation accuracy. A two-stage bearing estimation procedure is proposed. The initialisation stage involves spatial averaging over all suitable partial arrays. The refinement stage uses a local maximum-likelihood search. Typical radar model simulations and Cramér-Rao bound calculations demonstrate the efficiency of this approach compared with standard spatial smoothing using a uniform linear array.

1. INTRODUCTION

Despite the growing interest in arrayed sensor applications (such as direction finding and communications), few discussions have appeared about array geometries and corresponding signal processing algorithms capable of resolving multiple radar targets in bearing. This is partly due to sufficiently high Doppler frequency and range resolution which justifies the "single target per resolution cell" model usually adopted in radar studies. Nevertheless, some applications require spatial resolution of targets which are unresolved in range and Doppler, *eg*. the resolution in elevation angle of a sea-surface skimming object and its reflection.

In standard radar observation models, the optimum procedure is to use multi-dimensional temporal (matched) filtering prior to bearing estimation; this is the "single snapshot" model which has been extensively addressed for direction-of-arrival (DOA) estimation. Spatial smoothing [5] and over-determined Yule-Walker methods [7] have been proposed as suitable techniques, at least for the initialisation of some more sophisticated (nonlinear) maximum-likelihood (ML) estimation procedure. Unfortunately both these techniques are suitable only for uniform linear arrays (ULA's).

Our motivation for introducing a modified version of the spatial smoothing technique for nonuniform linear arrays (NLA's) is that given a fixed number of antenna elements M, sparse arrays have longer apertures than their corresponding ULA's and therefore have enhanced bearing estimation accuracies. Our approach is to design a class of NLA and simultaneously a corresponding signal processing algorithm, which is a "generalised spatial smoothing" (GSS) technique. We shall explore the limit bearing estimation accuracy of this class.

[*]This study was partly supported by the INTAS SASPARC grant.

2. DATA MODEL

In departing from the "single target per resolution cell" model, we consider the case where some rather small number of targets ($m \ll M$) are to be spatially resolved. The problem of bearing estimation for m targets irradiated at time t by the radar waveform $s(t)$, $t = 1, \dots, T$, and received by a linear antenna array consisting of M identical sensors can be reduced to estimation of the unknown bearing parameter $\theta = [\theta_1, \dots, \theta_m]^T$ in the equation

$$\mathbf{y}(t) \,=\, A(\theta)\,\mathbf{x}(t)\,s(t) \,+\, \mathbf{n}(t) \tag{1}$$

where $\mathbf{y}(t) \in \mathcal{C}^{M\times 1}$ is the vector of observed sensor outputs at time t (the "snapshot"), $A \in \mathcal{C}^{M\times m}$ is the transfer matrix, $\mathbf{x} \in \mathcal{C}^{m\times 1}$ is the vector of (unknown) complex target scattering coefficients, and $\mathbf{n}$ is additive noise. The targets are assumed to be non-fluctuating over the observation interval T. The classical "Swerling 1" model [1] incorporates a Gaussian distribution for the scattering coefficients. We assume all m targets have the same range and Doppler frequency in order to explore the potential of spatial-only resolution. The transfer matrix consists of the M-variate array manifold ("steering") vectors $A(\theta) = \left[\mathbf{a}_1, \dots, \mathbf{a}_m\right]$, where each column vector is

$$\mathbf{a}_j \,=\, \Big[1,\, \exp\left(i\pi d_2 \sin\theta_j\right),\, \dots,\, \exp\left(i\pi d_M \sin\theta_j\right)\Big]^T \tag{2}$$

where the array sensor positions are $\mathbf{d} = [d_1 \equiv 0, d_2, \dots, d_M]$ (measured in half-wavelength units). The noise $\mathbf{n}$ is assumed to have a complex Gaussian distribution with zero mean and power p_0^2, and to be spatially uncorrelated, possibly with temporal correlation given by $B(t_1, t_2)$. This assumption reflects the typical radar model of clutter broadly distributed in space mixed with white noise. We also assume that the scattering coefficients $\mathbf{x}(t)$ and the noise $\mathbf{n}(t)$ are uncorrelated for all t.

Clearly the model of Eqn. (1) consists of the standard radar model with fully correlated sources, as adopted in the field of DOA estimation [7], where we have sufficient statistics. Optimum temporal matched-filter processing is performed in all M channels to obtain the single snapshot $\hat{\mathbf{y}}$ which is then used for bearing estimation. Given initial bearing estimates, further refinement may be achieved by local ML-optimisation. One possible iterative approach stems from the following interpretation of the optimum ML algorithm obtained from the signal model with unknown powers p_j ($j = 1, \ldots, m$):

$$\frac{\mathcal{R}e\,\left(\dot{\mathbf{a}}_j^H R_j^{-1}\,\mathbf{a}_j\right)}{\mathbf{a}_j^H R_j^{-1}\,\mathbf{a}_j} = \frac{\mathcal{R}e\,\left(\hat{\mathbf{y}}^H R_j^{-1}\,\mathbf{a}_j\,\dot{\mathbf{a}}_j^H R_j^{-1}\,\hat{\mathbf{y}}\right)}{\left|\hat{\mathbf{y}}^H R_j^{-1}\,\mathbf{a}_j\right|^2} \tag{3}$$

where $\dot{\mathbf{a}}_j = \frac{\partial \mathbf{a}}{\partial \theta}\Big|_{\theta=\theta_j} = \mathrm{diag}[\mathbf{d}]\;\mathbf{a}_j$, and

$$R_j = \left(\mathbf{s}^H B^{-1}\,\mathbf{s}\right) I + \left(\mathbf{s}^H B^{-1}\,\mathbf{s}\right)^2 \sum_{k=1,\;k\neq j}^{m} \mathbf{a}_k\,\mathbf{a}_k^H\,p_k \tag{4}$$

where $\mathbf{s} = [s(t_1), \ldots, s(T)]^T$. If the parameters (θ_k, p_k), $k = 1, \ldots, \ell-1, \ell+1, \ldots, m$ of the "interfering" targets are precisely known, then Eqn. (3) coincides with the well-known estimation procedure for non-white (spatial) noise with known covariance [2]. This procedure can therefore be used in simulation studies to define the upper bound of realistically achievable accuracy when the true target parameters are unknown. In fact, such an upper bound is more realistic even than the Cramér-Rao bound (CRB) since for a single snapshot, a limited number of antenna sensors and a finite SNR, asymptotic conditions are unlikely to be met.

When initial estimates $\left(\hat{\theta}_k^{(0)}, \hat{p}_k^{(0)}\right)$, $k = 1, \ldots, m$ are available, Eqns. (3) and (4) may be used to construct an iterative scheme to calculate $\left(\hat{\theta}_\ell^{(n+1)}, \hat{p}_\ell^{(n+1)}\right)$ using the current set of estimates:

$$\begin{array}{lll} \left(\hat{\theta}_k^{(n+1)}, \hat{p}_k^{(n+1)}\right) & \text{for} \quad k = 1, \ldots, \ell-1 & \text{and} \\ \left(\hat{\theta}_j^{(n)}, \hat{p}_j^{(n)}\right) & \text{for} \quad j = \ell+1, \ldots, m. & \end{array} \tag{5}$$

A one-dimensional search, or even the direct analytic expression based on the first-order expansion, may be used to find the solution of the ML equation (3) at each iteration [2].

3. PARTIAL ARRAYS

We define a *partial array* to be a group of nonuniform linear non-contiguous sub-arrays of identical co-array structure. For example, the NLA

$$\mathbf{d}_{eg} = [0,\ 1,\ 5,\ 6,\ 9,\ 11,\ 12] \tag{6}$$

has the co-array (*ie*. set of consecutive inter-element differences)

$$\mathbf{c}_{eg} = [1,\ 4,\ 1,\ 3,\ 2,\ 1] \tag{7}$$

in which is embedded the partial array with co-array structure

$$\mathbf{c}_1 = [1,\ 5] \tag{8}$$

since this co-array repeatedly occurs as a fixed sub-array pattern of the original array elements as follows:

$$\begin{array}{l} \mathbf{d}_{11} = [0,\ 1,\ 6] \\ \mathbf{d}_{12} = [5,\ 6,\ 11] \\ \mathbf{d}_{13} = [0,\ 5,\ 6] \\ \mathbf{d}_{14} = [6,\ 11,\ 12]. \end{array} \tag{9}$$

We say that this partial array has a *multiplicity* (number of occurrences or *instances*) of $\kappa_1 = 4$. Note that the instances $\mathbf{d}_{13}$ and $\mathbf{d}_{14}$ exist as mirror-images (the co-array order is reversed). We define the *order* (ℓ) of a partial array to be the number of co-array elements involved. This partial array has order $\ell_1 = 2$, and aperture $a_1 = \sum_{j=1}^{\ell_1} \mathbf{c}_{1j} = 6$. The exhaustive list of partial arrays for the $\mathbf{d}_{eg}$ geometry is

$$\begin{array}{llll} \mathbf{c}_1 = [1,\ 5] & \kappa_1 = 4 & \ell_1 = 2 & a_1 = 6 \\ \mathbf{c}_2 = [1,\ 4] & \kappa_2 = 2 & \ell_2 = 2 & a_2 = 5 \\ \mathbf{c}_3 = [1,\ 6] & \kappa_3 = 2 & \ell_3 = 2 & a_3 = 7 \\ \mathbf{c}_4 = [1,\ 10] & \kappa_4 = 2 & \ell_4 = 2 & a_4 = 11 \\ \mathbf{c}_5 = [1,\ 11] & \kappa_5 = 2 & \ell_5 = 2 & a_5 = 12 \\ \mathbf{c}_6 = [5,\ 6] & \kappa_6 = 3 & \ell_6 = 2 & a_6 = 11 \\ \mathbf{c}_7 = [1,\ 5,\ 5] & \kappa_7 = 2 & \ell_7 = 3 & a_7 = 11 \\ \mathbf{c}_8 = [1,\ 5,\ 6] & \kappa_8 = 2 & \ell_8 = 3 & a_8 = 12 \end{array} \tag{10}$$

so that the number of embedded partial arrays is $n = 8$, with a total of $N = \sum_{j=1}^{n} \kappa_j = 19$ instances.

The spatial smoothing idea may be applied to a NLA providing it yields at least one partial array of multiplicity $\kappa \geq m$ and order $\ell \geq m$. The effectiveness of GSS is directly related to the number, variety and $\kappa\ell a$-properties of the available partial arrays in the following ways.

Firstly, we desire a NLA which contains as many partial arrays as possible for a given M, preferably each with large κ and ℓ. Therefore neither nonredundant nor minimum-redundancy arrays are suitable; we need to identify a new class of NLA.

Secondly, since a partial array is nonuniform in geometry, it admits manifold ambiguity (because $a > \ell$) [4]. Thus if only one partial array exists, there might be situations defined by the corresponding ambiguity generator sets [4] where standard MUSIC fails to provide unambiguous bearing estimates. Since different partial arrays have (generally) different ambiguity sets, by increasing the number of different partial arrays we decrease the probability of a combined ambiguous scenario.

Thirdly, recall that the main motivation for using nonuniform arrays is to increase the resolution capability beyond the uniform array limit. Evidently the partial arrays should have an aperture which is as long as possible, and in any case exceeds that of the corresponding uniform array.

These criteria have been used in a specifically designed optimisation procedure which begins with a nonredundant array of the desired aperture, then adds single array elements up to the desired number M in vacant sensor positions so as to maximise a generalised criterion which encompasses these three important features. While the details of this process appear in a separate paper [6], the array geometry optimisation results are illustrated by the following two examples of 16-element NLA.

A selection criterion (designed to resolve up to three targets) which involves the number and distribution of partial array apertures results in the following solution:

$$\mathbf{d}_{55} = [0,\ 1,\ 5,\ 6,\ 8,\ 10,\ 19,\ 23,\ 26,\ 34,\ 37,\ 41,\ 44,\ 52,\ 53,\ 55]. \tag{11}$$

Table 1 shows the $\kappa\ell$-distribution and Fig. 1 illustrates the a-distribution of partial arrays for this NLA. We expect this array to perform better than the 16-element ULA because of the large numbers of embedded partial arrays, each of significant aperture. The minimum-redundancy array of comparable total aperture ($M_\alpha = 58$) has 13 elements [3], so we could consider $\mathbf{d}_{55}$ to be a type of "optimal" solution by the introduction of only three additional elements to the minimum-redundancy structure.

A 16-element NLA with different partial array characteristics

	$\kappa=3$	$\kappa=4$	$\kappa=5$	$\kappa=6$	$\kappa=7$	$\kappa=8$	$\kappa=9$
$\ell=3$	33	9	0	0	0	0	0
$\ell=4$	32	3	0	0	0	0	0
$\ell=5$	12	0	0	0	0	0	0

Table 1: Partial array distribution by multiplicity (κ) and order (ℓ) for the NLA $\mathbf{d}_{55}$. For $1 \le c \le 18$ there are $n = 89$ partial arrays with a total of $N = 279$ instances available for spatial smoothing.

	$\kappa=3$	$\kappa=4$	$\kappa=5$	$\kappa=6$	$\kappa=7$	$\kappa=8$	$\kappa=9$
$\ell=3$	76	26	2	0	0	0	0
$\ell=4$	43	0	0	0	0	0	0
$\ell=5$	3	0	0	0	0	0	0

Table 2: Partial array distribution for the NLA $\mathbf{d}_{34}$, where $n = 150$ and $N = 480$ for $1 \le c \le 18$.

is illustrated by the shorter array

$$\mathbf{d}_{34} = [0,\ 1,\ 4,\ 5,\ 8,\ 9,\ 10,\ 14,\ 15,\ 16,\ 18,\ 22,\ 23,\ 25,\ 32,\ 34]. \quad (12)$$

Table 2 and Fig. 2 show its partial array $\kappa\ell a$-properties. While the maximum partial array aperture ($\max(a) = 24$) is here less than that of $\mathbf{d}_{55}$, the number of partial arrays is much greater. In any case, we would expect $\mathbf{d}_{34}$ to perform better than the 16-element ULA. Note that the 10-element optimal minimum-redundancy array (which contains no suitable partial arrays) has a similar aperture ($M_\alpha = 36$) [3], hence all $N = 480$ instances are created by the addition of six antenna sensors.

Obviously the bearing estimation performance achieved should be compared to that of a 16-element ULA. A standard spatial smoothing technique here is to use the single partial array $\mathbf{c} = [1,\ \ldots,\ 1]$ of order $\ell = 14$, which has multiplicity $\kappa = 6$ (three of them mirror-images) and aperture $a = 14$.

Note that our embedded partial array distributions are found by exhaustive computer search. Due to a "combinatorial explosion", these searches can only be conducted over a finite range of values of ℓ and c (the possible value of any partial co-array element). The partial array distributions we present here are standardised on what is very likely to be completeness for these scenarios: $3 \le \ell \le 5$ and $1 \le c \le 18$.

These examples illustrate a range of possible NLA trade-offs, where apertures are maximised by redundancy minimisation and partial array distributions are optimised by creating redundancies.

4. PARTIAL ARRAY MUSIC TECHNIQUE FOR BEARING ESTIMATION INITIALISATION

Suppose that an NLA yields a total of N partial arrays, each of multiplicity κ_i, order ℓ_i and aperture a_i ($i = 1, \ldots, N$). Let $\mathbf{y}_{ij}$ be a $(\ell_i + 1)$-variate snapshot vector corresponding to the j^{th} instance ($j = 1, \ldots, \kappa_i$) of the i^{th} partial array. If any instance of a partial array occurs as a mirror-image (*ie*. in reverse order), then the corresponding snapshot vector is observed by reversing the order of antenna samples and taking the complex conjugate of the vector. Thus for each partial array we may define the $(\ell_i + 1) \times (\ell_i + 1)$ partial array covariance matrix by spatial smoothing to be

$$\hat{R}_i = \sum_{j=1}^{\kappa_i} \mathbf{y}_{ij}\, \mathbf{y}_{ij}^H. \quad (13)$$

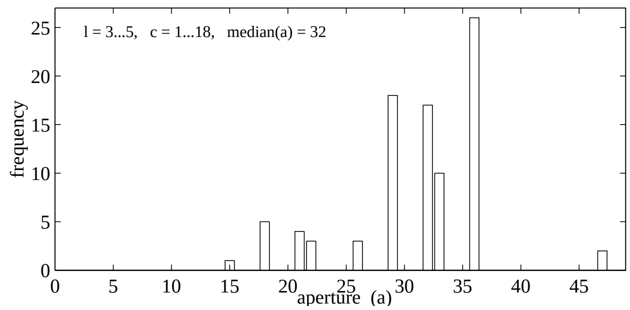

Figure 1: Aperture histogram of partial arrays embedded in $\mathbf{d}_{55}$.

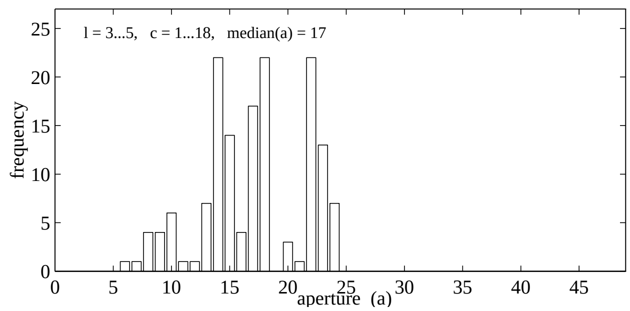

Figure 2: Aperture histogram of partial arrays embedded in $\mathbf{d}_{34}$.

Let $\hat{G}_i$ be the noise eigen-subspace of $\hat{R}_i$, then $\hat{G}_i$ consists of at least one eigenvector (since $m \ll M$).

We may now introduce the PA-MUSIC technique as follows:

$$\text{find} \quad \max_\theta f_{PA}(\theta) \;:=\; \min_\theta \sum_{i=1}^{N} \mathbf{a}_i^H(\theta)\, \hat{G}_i\, \hat{G}_i^H\, \mathbf{a}_i(\theta) \quad (14)$$

where $\mathbf{a}_i(\theta)$ is the $(\ell_i + 1)$-variate manifold vector which corresponds to the given partial array geometry. Evidently this approach eliminates non-coinciding ambiguities.

5. RESULTS OF NUMERICAL SIMULATIONS

The GSS procedure consists of an initialisation step followed by local ML refinement. The initialisation step is based on the PA-MUSIC approach involving all appropriate partial arrays. In order to demonstrate the trade-off between array aperture and number of partial array instances involved in spatial averaging (N), the same three-target scenario is tested for bearing estimation accuracy by the three different arrays $\mathbf{d}_{15}$ (the 16-element ULA), $\mathbf{d}_{34}$ and $\mathbf{d}_{55}$. Summary statistics for these arrays are as follows:

$$\begin{array}{lll} \text{median}(a_{15}) = 14, & n_{15} = 1, & N_{15} = 6 \\ \text{median}(a_{34}) = 17, & n_{34} = 150, & N_{34} = 480 \\ \text{median}(a_{55}) = 32, & n_{55} = 89, & N_{55} = 279 \end{array} \quad (15)$$

The signal-to-noise ratio is equal for all three targets. The Cramer-Rao bound is calculated for each scenario, though it is not expected to be a very tight bound in these cases, as we have mentioned. Hence we supplement each trial by simulation of the exact ML procedure of Eqns. (3) and (4) with the exact parameters (θ_i, p_i) for the two other "interfering" targets. Each experiment consists of 500 independent Monte-Carlo trials simulating independent realisations of single-snapshot stochastic data, whose principal results are the mean relative error (bias) and root-mean-square

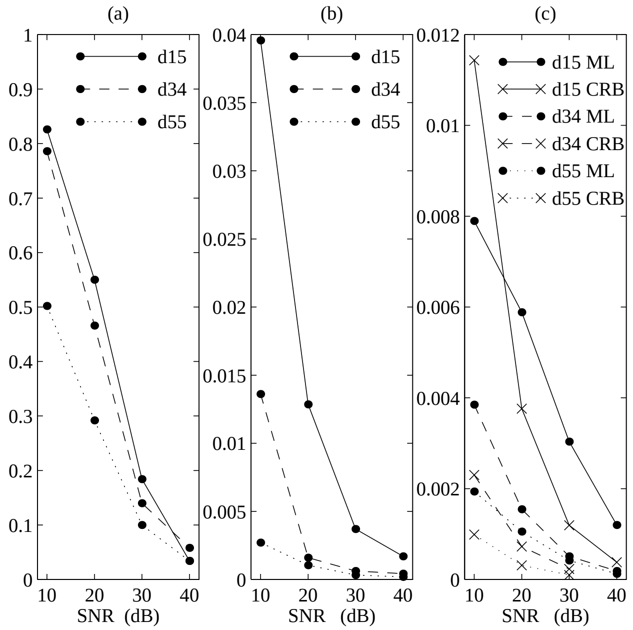

Figure 3: (a) Abnormal trial probability as a function of SNR for various NLA geometries, (b) maximum refined bearing estimation accuracy (excluding abnormal trials), (c) maximum Cramér-Rao bound and maximum exact ML total error.

error (standard deviation) for the estimated bearings. The PA-MUSIC routine operates on a spatial frequency grid with spacing $\Delta w = 4.88 \times 10^{-4}$, while the ML refinement algorithm was implemented on an eight times finer grid. (We measure spatial frequency in terms of $w = \sin\theta$.)

Since the main purpose of the PA-MUSIC step is initialisation for ML-refinement, in addition to the bias and standard deviation its performance is measured by the percentage of "abnormal" trials. An abnormal trial occurs when PA-MUSIC fails to properly resolve close targets and selects a completely inappropriate peak in the MUSIC pseudo-spectrum. The mechanism for such failures is well known.

In accordance with the main idea of our approach, we investigate several different super-resolution situations where the spatial frequencies of the $m = 3$ sources are set at $w = [-0.5,\ 0,\ 0.05]$.

Fig. 3(a) presents the sample probability of abnormal trials (failed target resolution) as a function of SNR for all three geometries. By the conventional beamwidth resolution limit formula $w_{CBF} = 2/a$, almost all partial arrays under consideration have a beamwidth greater than the source separation $\Delta = 0.05$; $\mathbf{d}_{55}$ is the only geometry that has any partial arrays with aperture exceeding $a_{crit} = 2/\Delta \simeq 40$. Hence it is not surprising that for this close-source separation, all arrays have a high probability of abnormals, especially at low SNR's, but that $\mathbf{d}_{55}$ is the best initialiser. Indeed the probability of correct initialisation is, as we expect, directly related to the partial array aperture distribution, with $\mathbf{d}_{15}$ performing the worst.

Fig. 3(b) shows the maximum refined total error (excluding abnormal trials) resulting from PA-MUSIC plus local ML refinement. We see that $\mathbf{d}_{55}$ is more accurate than $\mathbf{d}_{34}$, which in turn is more accurate than $\mathbf{d}_{15}$. Here the PA-MUSIC accuracy over the set of normal trials is in direct relation to the length of partial array apertures, rather than the number of them. These final accuracy results repeat the same behaviour as for the unrefined results.

Finally, it is interesting to compare the final GSS accuracy with the corresponding Cramer-Rao bounds and the exact ML total error of Eqns. (3) and (4) obtained during simulation (Fig. 3(c)). We see that in general the ML bound is significantly larger than the CRB. As was discussed previously, the CRB is not a tight enough bound in this pre-asymptotic scenario. We also see that the difference between the actual final accuracy achieved by GSS on $\mathbf{d}_{55}$ and the exact ML bound is almost negligible.

6. SUMMARY

We have introduced a new approach for bearing estimation of multiple radar targets which are unresolved in range and in Doppler frequency, where the number of such targets is much smaller than the number of antenna sensors. This applies to the single snapshot signal model, and in fact in all cases of rank-1 signal covariance, as in the case of coherent signals. This approach involves the design of nonuniform linear array geometries *and* the corresponding design of a suitable signal processing algorithm, which we have called the generalised spatial smoothing (GSS) algorithm.

This defines a special class of NLA which incorporates a trade-off between the maximum array aperture that can be achieved with a particular number of sensors, and the maximum number and quality of "partial arrays" (groups of noncontiguous sub-arrays of identical co-array structure) necessary for spatial averaging.

The associated bearing estimation algorithm consists of two stages. Firstly, the initialisation stage adopts spatial smoothing over all available suitable partial arrays to form sample covariance matrices and apply MUSIC-type techniques. Secondly, the refinement of bearing estimates takes place via a local one-dimensional maximum-likelihood iterative optimisation.

For the situations examined here with successful initialisation, a practically optimal final bearing estimation accuracy is demonstrated, one which is significantly superior to the conventional ULA geometry complemented by standard spatial smoothing algorithm.

7. REFERENCES

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