PERFORMANCE OF OTH RADAR ARRAY CALIBRATION

I.S.D.Solomon, Yu.I.Abramovich, D.A.Gray and S.J.Anderson

Defence Science and Technology Organisation, P.O.Box 1500, Salisbury 5108, Australia
CRC for Sensor Signal and Information Processing
University of Adelaide
University of South Australia

ABSTRACT

Array calibration algorithms for over-the-horizon (OTH) radar arrays have been recently proposed in the literature. These algorithms perform array calibration by using echoes from ionised meteor trails, and estimate sensor position errors and mutual coupling. In this paper we derive the Cramer-Rao performance bound for this array calibration problem and then investigate the performance bound. We obtain insight on the achievable accuracy as a function of the signal-to-noise ratio, number of snapshots and number of sources. Further, we consider the advantage of using sources with known bearings, as opposed to unknown bearings, and consider the identifiability of the array calibration problem.

1. INTRODUCTION

Array calibration algorithms for over-the-horizon (OTH) radar arrays have been proposed in [10] [11]. These algorithms estimate the sensor position errors and mutual coupling, using external sources. Array calibration in [10] is performed by exploiting echoes from ionised meteor trails, while in [11] a variety of sources (including echoes from ionised meteor trails) are used for calibrating the radar array.

In practice the important case is where only disjoint echoes from ionised meteor trails are used, in either [10] or [11], since these sources have very attractive properties from an array calibration perspective [12]. By disjoint we mean the echoes do not occupy both the same time snapshots and the same radar range cells. In this paper we obtain insight into this important practical case by analysing the Cramer Rao lower bound (CRLB). The array calibration problem can only be solved when the CRLB exist, hence the existence of the CRLB addresses the question of identifiability. Further, since the CRLB gives the minimum variance that an unbiased estimator can obtain, it gives the accuracy attainable under given scenarios.

The CRLB for array calibration has been considered in [8] [14] [2] [1] [6] [7] [9]. Only [8] and [7] consider multiple sources which are either temporally or spatially disjoint. In [8] only sensor positions are considered, and prior statistics are used in the derivation of the CRLB. In [7] the source directions-of-arrival (DOAs) are assumed known a priori and the coupling matrix is unstructured. Here we determine the CRLB for disjoint sources where the sensor positions, symmetric coupling matrix, and source DOAs are unknown.

2. PROBLEM FORMULATION

For a narrowband single-mode signal impinging an M element array, in the absence of mutual coupling, the output of the mth sensor is

$$z_{m1}(t) = (1+\alpha_m)e^{-j\phi_m} s_1(t) e^{-jw\tau_{m1}} + n_m(t) \qquad (1)$$

where α_m & ϕ_m are the receiver gain and phase errors, $s_1(t)$ is the received signal, and $n_m(t)$ is additive receiver noise. The radar operating frequency is w, $\tau_{mn} = (x_m \sin\theta_n + y_m \cos\theta_n)/v$, x_m & y_m define the position of the mth sensor, θ_1 is the DOA of the signal (with respect to broadside), and v is the speed of light in free space.

The vector of M sensor outputs, from the array, is

$$\mathbf{z}_1(t) = \mathbf{\Gamma}\mathbf{a}(\theta_1)s_1(t) + \mathbf{n}(t) \qquad (2)$$

where $\mathbf{z}_1(t) = [z_{11}(t), z_{21}(t), ..., z_{M1}(t)]^T$,
$\mathbf{\Gamma} = diag\{(1+\alpha_1)e^{-j\phi_1}, ..., (1+\alpha_M)e^{-j\phi_M}\}$,
$\mathbf{a}(\theta_n) = [e^{-jw\tau_{1n}}, e^{-jw\tau_{2n}}, ..., e^{-jw\tau_{Mn}}]^T$,
$\mathbf{n}(t) = [n_1(t), n_2(t), ..., n_M(t)]^T$.
In the presence of mutual coupling [3]

$$\mathbf{z}_1(t) = \mathbf{C}\mathbf{\Gamma}\mathbf{a}(\theta_1)s_1(t) + \mathbf{n}(t) \qquad (3)$$

where $\mathbf{C} = (\mathbf{I}_M + \mathbf{Z}_O/Z_L)^{-1}$ is called the coupling matrix. Matrix $\mathbf{I}_M$ is the MxM identity matrix, $\mathbf{Z}_O$ is the array mutual coupling matrix, and Z_L is the scalar load impedance. The covariance matrix for this signal, assuming zero mean noise, is

$$\mathbf{R}_1 = E\{\mathbf{z}_1(t)\mathbf{z}_1(t)^H\} \qquad (4)$$

Now consider N disjoint echoes. Most meteor trail echoes we have observed are resolvable in time and range, indicating different underlying physical mechanisms, and hence we represent these as statistically disjoint sources [12]. The covariance matrix for the nth disjoint echo

$$\mathbf{R}_n = E\{\mathbf{z}_n(t)\mathbf{z}_n(t)^H\} \qquad (5)$$

where $\mathbf{z}_n(t)$ is the vector of M sensor outputs for the nth disjoint echo.

The problem is then to estimate the sensor positions and coupling matrix, given the N covariance matrices. We have assumed here that the data $\mathbf{z}_n(t)$ have been corrected for the receiver gain/phase errors, using internal calibration.

Note, the algorithm in [10] is based on minimising the cost function

$$Q = \sum_{n=1}^{N} \|\mathbf{U}(n)^H \mathbf{C}\mathbf{a}(\theta_n)\|^2 \tag{6}$$

where $\mathbf{U}(n)$ is the matrix whose columns are the eigenvectors of $\mathbf{R}_n$ which correspond to the noise eigenvalues of $\mathbf{R}_n$, and N is the number of disjoint echoes. The algorithm in [11], for this special case of disjoint echoes, is based on minimising

$$Q = \sum_{n=1}^{N} \|\mathbf{e}_n - \mathbf{C}\mathbf{a}(\theta_n)s_n\|^2 \tag{7}$$

where $\mathbf{e}_n$ is the eigenvector corresponding to the signal eigenvalue (principal eigenvector) of $\mathbf{R}_n$ and s_n is a complex scalar.

3. CRAMER RAO LOWER BOUND

The likelihood function for the complete data set $\{\mathbf{z}_n(t),\ t = 1, 2, ..., T\ \&\ n = 1, 2, ...N\}$ is given by

$$p[\mathbf{z}(1), \mathbf{z}(2), ..., \mathbf{z}(T)/\mathbf{\Psi}] = \prod_{t=1}^{T} \frac{1}{\pi^M \|\mathbf{R}\|} \exp(-\mathbf{z}(t)^H \mathbf{R}^{-1} \mathbf{z}(t)) \tag{8}$$

where $\mathbf{z}(t) = [\mathbf{z}_1(t)^T, \mathbf{z}_2(t)^T, ..., \mathbf{z}_N(t)^T]^T$, $\mathbf{\Psi} = [\theta, \mathbf{x}, \mathbf{y}, \mathbf{C}]$ and the MNxMN matrix $\mathbf{R}$ is

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{R}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{R}_N \end{bmatrix} \tag{9}$$

where the exact covariance matrix of the nth echo/source is $\mathbf{R}_n = \sigma_s^2 \mathbf{C}\mathbf{a}(\theta_n)\mathbf{a}(\theta_n)^H \mathbf{C}^H + \sigma_n^2 \mathbf{I}_M$, assuming all echoes have signal-to-noise ratio (SNR) = σ_s^2/σ_n^2. The unconditional CRLB [13] is then

$$CRLB(\mathbf{\Psi}) = [J_{kl}]^{-1} \tag{10}$$

where the elements of the symmetric Fisher Information Matrix (FIM) are

$$J_{kl} = J_{lk} = T tr\{\mathbf{R}^{-1} \partial\mathbf{R}/\partial\Psi_k \mathbf{R}^{-1} \partial\mathbf{R}/\partial\Psi_l\} \tag{11}$$

where $tr\{\}$ is the trace operator. Since $\mathbf{R}$, $\mathbf{R}^{-1}$, $\partial\mathbf{R}/\partial\Psi_k$ are all block structured matrices

$$J_{kl} = T \sum_{n=1}^{N} tr\{\mathbf{R}_n^{-1} \partial\mathbf{R}_n/\partial\Psi_k \mathbf{R}_n^{-1} \partial\mathbf{R}_n/\partial\Psi_l\} \tag{12}$$

The elements of the FIM are now given (see [10] for the details).

Since we have disjoint sources, $\partial\mathbf{R}_n/\partial\theta_l$ is a zero matrix for $n \neq l$, and hence J_{θ_k,θ_l} is zero for $k \neq l$. Now

$$J_{\theta_k,\theta_k} = 2T{\sigma_s}^4 \Re\{tr\{\mathbf{R}_k^{-1} \mathbf{C}\dot{\mathbf{a}}_{\theta_k}(\theta_k)\mathbf{a}(\theta_k)^H \mathbf{C}^H \mathbf{R}_k^{-1} \mathbf{C} \\ (\dot{\mathbf{a}}_{\theta_k}(\theta_k)\mathbf{a}(\theta_k)^H + \mathbf{a}(\theta_k)\dot{\mathbf{a}}_{\theta_k}(\theta_k)^H)\mathbf{C}^H\}\} \tag{13}$$

where $\dot{\mathbf{a}}_{\theta_k}(\theta_k) = \partial\mathbf{a}(\theta_k)/\partial\theta_k = \mathbf{D}(\theta_k)\mathbf{a}(\theta_k)$ (with $\mathbf{D}(\theta_k) = (-2\pi j/\lambda) diag\{\mathbf{x}\cos(\theta_k) - \mathbf{y}\sin(\theta_k)\}$), and $\Re\{\}$ is the real part.

$$J_{x_k,x_l} = 2T{\sigma_s}^4 \sum_{n=1}^{N} \Re\{tr\{\mathbf{R}_n^{-1} \mathbf{C}\dot{\mathbf{a}}_{x_k}(\theta_n)\mathbf{a}(\theta_n)^H \mathbf{C}^H \mathbf{R}_n^{-1} \\ \mathbf{C}(\dot{\mathbf{a}}_{x_l}(\theta_n)\mathbf{a}(\theta_n)^H + \mathbf{a}(\theta_n)\dot{\mathbf{a}}_{x_l}(\theta_n)^H)\mathbf{C}^H\}\} \tag{14}$$

where $\dot{\mathbf{a}}_{x_k}(\theta_n) = \partial\mathbf{a}(\theta_n)/\partial x_k = \mathbf{dx}_k \odot \mathbf{a}(\theta_n)$, $\odot$ is the Hadamard product, and $\mathbf{dx}_k$ is an M element row vector with all but the kth element zero; the kth element is $(-2\pi j/\lambda)\sin(\theta_n)$, where λ is the radar wavelength. Similarly for the y coordinate and x-y coordinate terms.

$$J_{c_k,c_l} = 2T{\sigma_s}^4 \sum_{n=1}^{N} \Re\{tr\{\mathbf{R}_n^{-1} \dot{\mathbf{C}}_{c_k}\mathbf{a}(\theta_n)\mathbf{a}(\theta_n)^H \mathbf{C}^H \mathbf{R}_n^{-1} \\ (\dot{\mathbf{C}}_{c_l}\mathbf{a}(\theta_n)\mathbf{a}(\theta_n)^H \mathbf{C}^H + \mathbf{C}\mathbf{a}(\theta_n)\mathbf{a}(\theta_n)^H \dot{\mathbf{C}}_{c_l}^H)\}\} \tag{15}$$

where matrix $\dot{\mathbf{C}}_{c_k} = \partial\mathbf{C}/\partial c_k$, c_k being the amplitude/phase of an element in the symmetric coupling matrix.

$$J_{\theta_k,x_l} = T{\sigma_s}^4 tr\{\mathbf{R}_k^{-1}\mathbf{C}(\dot{\mathbf{a}}_{\theta_k}(\theta_k)\mathbf{a}(\theta_k)^H + \\ \mathbf{a}(\theta_k)\dot{\mathbf{a}}_{\theta_k}(\theta_k)^H)\mathbf{C}^H \mathbf{R}_k^{-1}\mathbf{C} \\ (\dot{\mathbf{a}}_{x_l}(\theta_k)\mathbf{a}(\theta_k)^H + \mathbf{a}(\theta_k)\dot{\mathbf{a}}_{x_l}(\theta_k)^H)\mathbf{C}^H\} \tag{16}$$

$$J_{\theta_k,c_l} = T{\sigma_s}^4 tr\{\mathbf{R}_k^{-1}\mathbf{C}(\dot{\mathbf{a}}_{\theta_k}(\theta_k)\mathbf{a}(\theta_k)^H + \\ \mathbf{a}(\theta_k)\dot{\mathbf{a}}_{\theta_k}(\theta_k)^H)\mathbf{C}^H \mathbf{R}_k^{-1} \\ (\dot{\mathbf{C}}_{c_l}\mathbf{a}(\theta_k)\mathbf{a}(\theta_k)^H \mathbf{C}^H + \mathbf{C}\mathbf{a}(\theta_k)\mathbf{a}(\theta_k)^H \dot{\mathbf{C}}_{c_l}^H)\} \tag{17}$$

$$J_{c_k,x_l} = T{\sigma_s}^4 \sum_{n=1}^{N} tr\{\mathbf{R}_n^{-1}(\dot{\mathbf{C}}_{c_k}\mathbf{a}(\theta_n)\mathbf{a}(\theta_n)^H \mathbf{C}^H + \\ \mathbf{C}\mathbf{a}(\theta_n)\mathbf{a}(\theta_n)^H \dot{\mathbf{C}}_{c_k}^H)\mathbf{R}_n^{-1}\mathbf{C} \\ (\dot{\mathbf{a}}_{x_l}(\theta_n)\mathbf{a}(\theta_n)^H + \mathbf{a}(\theta_n)\dot{\mathbf{a}}_{x_l}(\theta_n)^H)\mathbf{C}^H\} \tag{18}$$

4. PERFORMANCE EVALUATION

We studied the performance for a nominal uniform linear 4-element array, with inter-element spacing of $d = 0.4\lambda$. The randomly generated sensor position errors in the x-coordinate (along the array) and y-coordinate (perpendicular to array) are given in Table 1. The coupling matrix we employed was experimentally measured from the Jindalee OTH radar transmitter array [5]. Unless specified otherwise, 10 sources equally spaced from 0 to 180 degrees in azimuth, each with SNR of 30dB and 500 snapshots were considered (as considered in [10]).

Sensor	1	2	3	4
y-coordinate	-0.1149	0.0674	0.0011	-0.0361
x-coordinate	-0.0499	0.0130	0.0156	-0.0162

Table 1: Position Errors (λ)

Since in [10] [11] it is assumed that the location of one sensor and the direction to another sensor is known there are only 5 sensor position parameters $(2M - 3)$ which are unknown. For the coupling parameters, since we assumed a symmetric coupling matrix and place the constraint that $c_{11} = 1$, 18 coupling values $(M(M+1) - 2)$ are unknown. Finally since all DOAs are unknown, for 10 sources 10 DOAs (N) are unknown. The total number of unknown parameters is then 33 (the noise power is assumed to be known).

Figure 1 shows the CRLB for the estimation of the third sensor's y-coordinate $(-)$ and x-coordinate $(--)$, and the coupling value c_{12}'s amplitude $(-.-)$ and phase $(...)$. The standard deviation (STD) variation with SNR, for a range of meteor echo SNR's, are shown for 500 snapshots (the STD values for 10 snapshots are simply 50 times larger). Note the STD values for the sensor positions are in units of wavelengths, while the coupling value phase is in units of radians. The monotonically decreasing STD values with SNR indicates the problem is well defined. The STD values indicate that good array calibration accuracy can be achieved for 10 echoes with their typical 20-30dB SNR and typical number of snapshots (5-15). As expected, the calibration accuracy achievable for the third sensor's x-coordinate and y-coordinate position are similar.

The STD variation with number of snapshots is shown in figure 2. While the number of snapshots for meteor echoes is typically less than about 15-20, the behaviour for higher values gives useful insight. The variation observed is in accordance with the 1/T fall off expected, and indicates that if one can obtain more snapshots from meteor echoes, the performance can be improved. Note we expect each meteor echo snapshot to be independent, even though meteors are passive sources, since the scattering process is a rapidly time-varying process [4].

Figure 3 shows the STD variation with number of sources. The performance attainable increases rapidly initially and then improves more gradually. The FIM was non-invertible, and hence the problem non-identifiable, for less than four sources. One can conclude from this figure that the more sources one can use for array calibration the greater the achievable accuracy. However, since the curves start to flatten off for high number of sources, the improvement obtained by adding further sources becomes minimal.

It has been mentioned in [6] that better array calibration accuracy can be obtained by using "Active Array Calibration" (special sources with known DOAs) as compared with "Passive Array Calibration" (sources of opportunity with unknown DOAs). Figure 4 shows the CRLB of the third sensor's y-coordinate, for the standard case $(-)$ together with some important special cases. The dashed curve $(--)$ shows the performance achievable if the source DOAs are known a priori. The improvement for sensor position estimation is a factor of about 2.5-3, while for the coupling amplitude and phase (not shown) the improvement is a factor of about 10. Hence Active Array Calibration does perform better, but clearly the difference can be offset by the use of sources with higher SNR and number of snapshots (and to some extent by using a larger number of sources).

Also shown in figure 4 is the CRLB for the case where other combinations of parameters are known a priori. For sensor position estimation, knowing the coupling values a priori $(...)$ gives slightly better results than knowing the DOAs $(--)$, while as expected knowing both the coupling and DOAs a priori $(-.-)$ gives even better performance. For the estimation of c_{12}'s amplitude (not shown), the difference in performance between the three cases is small, with the best accuracy obtained with both known DOAs and sensor positions (as expected). In the case of c_{12}'s phase (not shown), knowing the sensor positions a priori is better than knowing the DOAs a priori, and again knowing both a priori achieves the best results.

Ionised meteor trails are formed when meteoriods enter the earths atmosphere and are at altitudes of about 100km. Hence, meteor trail echoes reach OTH radar arrays from heights of about 100km. Until now we have assumed that the sources are at zero elevation, but now we investigate how array calibration performance is effected by the elevation angle of meteor echoes. Figure 5 shows the achievable accuracy as a function of the range of the meteor echoes; the closer the sources are from the radar the higher elevation angle. The results clearly show that calibration accuracy is seriously effected for echoes from ranges less than about 100km, with the accuracy being independent of range for echoes from ranges greater than about 200km. For simplicity we have ignored the drop off in antenna gain with elevation and the decrease in meteor echo intensity with range.

We have observed that as the azimuth spread over which the sources exist increases, the performance attainable increases. It should be mentioned that even though we consider a linear array, once sensor position errors are introduced the array is no longer linear, and hence improvement in performance is expected when the source spread is above 180 degrees too (as we have observed).

5. CONCLUSION

The CRLB, for the problem of OTH radar array calibration using meteor trails echoes, has been presented. Simulations have been used to illustrate how the bound decreases with SNR, number of snapshots and number of sources. For a 4-element array considered, the array calibration problem was non-identifiable when less than four sources were used. "Active Array Calibration" was shown to produce better accuracy than "Passive Array Calibration", at the cost of requiring special sources. The influence of the elevation angle of meteor echoes on array calibration accuracy was also outlined.

6. REFERENCES

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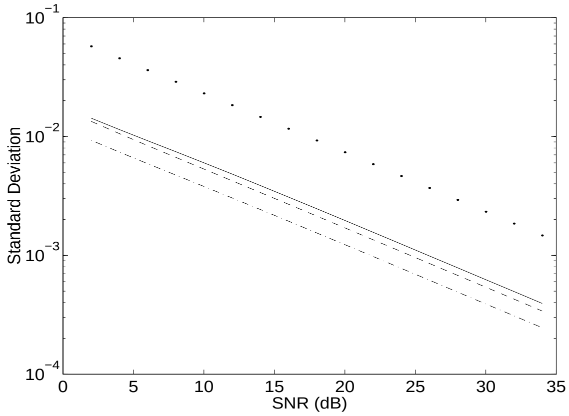

Figure 1: CRLB variation with SNR : third sensor's y-coordinate $(-)$ and x-coordinate $(--)$, and c_{12}'s amplitude $(-.-)$ and phase $(\ldots)$.

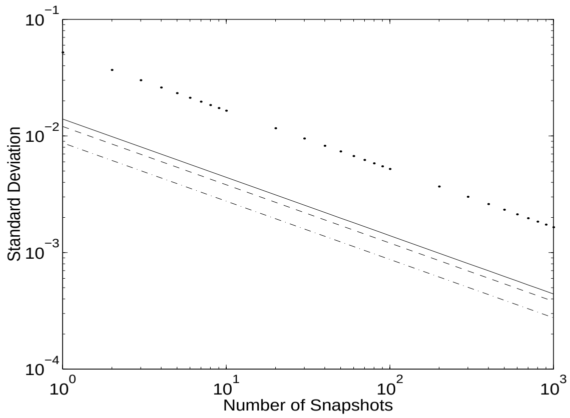

Figure 2: CRLB variation with number of snapshots : third sensor's y-coordinate $(-)$ and x-coordinate $(--)$, and c_{12}'s amplitude $(-.-)$ and phase $(\ldots)$.

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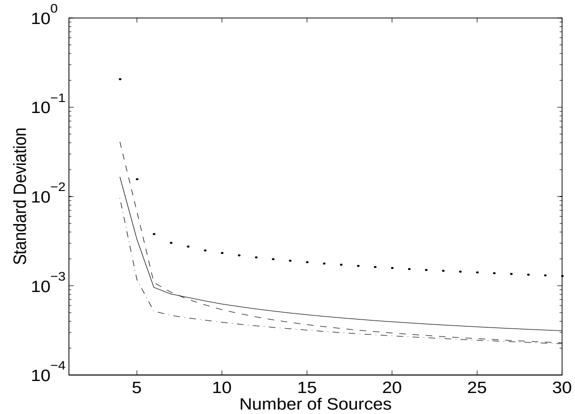

Figure 3: CRLB variation with number of sources : third sensor's y-coordinate $(-)$ and x-coordinate $(--)$, and c_{12}'s amplitude $(-.-)$ and phase $(\ldots)$.

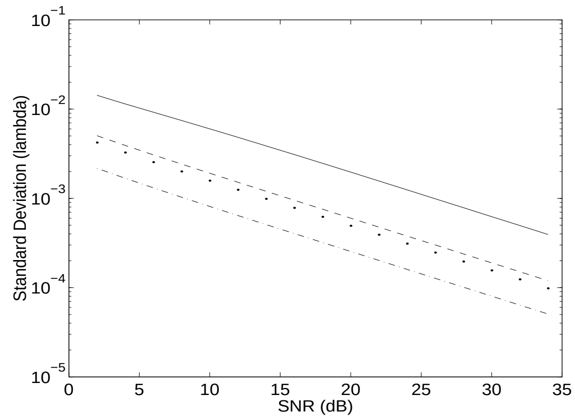

Figure 4: CRLB variation with SNR for the third sensor's y-coordinate : (a) DOAs and coupling values unknown $(-)$; (b) DOAs known but coupling values unknown $(--)$; (c) DOAs unknown but coupling values known $(\ldots)$; (d) DOAs and coupling values known $(-.-)$.

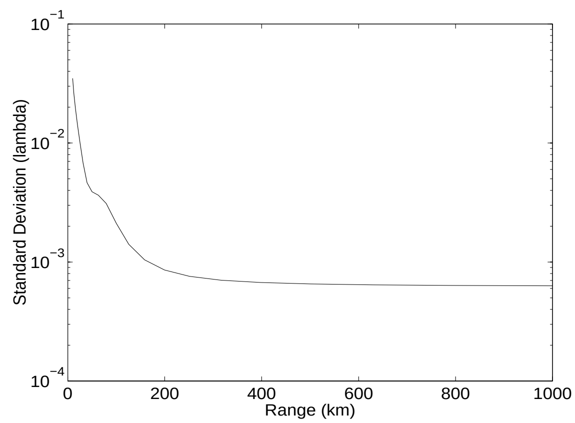

Figure 5: CRLB variation with range, for the third sensor's y-coordinate, when the elevation angle of meteor trail echoes is considered.