THE EFFECTIVE BANDWIDTH OF STABLE DISTRIBUTIONS

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ABSTRACT

In this paper the effective bandwidths of stable distributions are studied. Effective bandwidths are being heavily promoted as the most appropriate method for call admission control (CAC) and resource allocation within ATM networks. Recent work in teletraffic modelling has suggested that models based on stable distributions provide an efficient mechanism for capturing the long range dependence and infinite variance associated with teletraffic data (the Joseph and Noah effects; see [1]). This has potentially serious implications for effective bandwidths and we show how the effective bandwidth of such data is theoretically infinite. We then present two approximate methods for estimating the effective bandwidth of data based on stable distribution.

1. INTRODUCTION

The emergence of Broadband-ISDN networks has led to the development of the Asynchronous Transfer Mode; a networking paradigm capable of switching both time and loss sensitive services simultaneously. There is a need for resource allocation via a CAC scheme. A prerequisite of this is an accurate statistical characterisation of the traffic likely to be found on such networks and the impact of such traffic on proposed CAC schemes. This paper considers the impact of infinite variance processes on the effective bandwidth measure. The paper is structured as follows. Initially we introduce the effective bandwidth and show how it can be used in CAC. We then show the analytical result that the effective bandwidth measure of almost all stable distributions is infinite for realistic loss coefficients. However we go on to consider the practical implications and from this suggest two methods of estimating adapted forms of the effective bandwidth. Finally we present some simulation results which show that at present online estimation techniques work for infinite variance processes and that our adapted analytical estimates tend to be biased below. We conclude by suggesting why this is the case and identify how it can be corrected.

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2. EFFECTIVE BANDWIDTHS

The effective bandwidth of a VBR source is a value that lies somewhere between its mean and peak transmission rates and is related to the network performance by a loss coefficient [2]. Work reported in [3] and [4] suggest that they have potential for CAC on ATM.

However this measure is affected by the correlation structure of the source and a loss parameter that is chosen to match the QoS demands of that source. If the source is then serviced at its effective bandwidth it will conform to the QoS constraints imposed. In addition if several sources are serviced simultaneously at one switch and all are serviced at their effective bandwidths, then their QOS demands will not be violated [5].

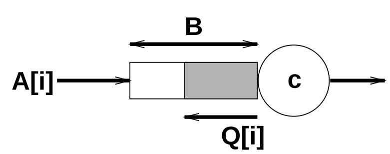

Figure 1: *A single server queue.* $A[i]$ *is a discrete arrival process,* c *is the service rate,* B *is the buffer size and* $Q[i]$ *is the size of the queue in the buffer.*

We can state this more formally. Consider the single server queue in Figure 1. The VBR video service is a discrete time stochastic arrival process, $A[i]$, and $Q[i]$ is the queue size of the buffer at time-slot i. If the queue is serviced at a constant rate, c then the queue-length at any time-slot can be calculated using the recursion

$$Q[i] = (Q[i-1] + A[i] - c, 0)^+, \qquad (1)$$

where $(a,b)^+$ is defined as the maximum out of a and b. Now define the scaled cumulant generating function

$$\Lambda(\theta) = \lim_{n\to\infty} \frac{1}{n} \log E[\exp(\theta(A[1] + \cdots + A[n]))]. \qquad (2)$$

If certain assumptions about the arrival process [5] (i.e. it is stationary and mixing) and the stability of the queue (i.e. $E(A[i]) < c$) are met then we can say

$$\lim_{B\to\infty} \frac{1}{B} \log P(Q[i] > B) = -\theta^*, \qquad (3)$$

where $\theta^* = \text{supremum}\{\theta > 0 : \Lambda(\theta) < \theta c\}$. We then define the effective bandwidth of the arrival process,

$$\delta(\theta) = \frac{\Lambda(\theta)}{\theta}. \qquad (4)$$

Exact expressions for the effective bandwidth are not available for realistic traffic sources. Several studies have been published which suggest that effective bandwidths have the potential to be applied to CAC schemes. For recent results on measurement based effective bandwidth estimation for CAC see the paper by Gibbens and Kelly [6].

3. THE EFFECTIVE BANDWIDTH OF STABLE DISTRIBUTIONS

In this section we consider the effective bandwidths of stable distributions and stable models [7]. The first result is disheartening as it proves that all stable distributions (apart from the Gaussian) have an infinite effective bandwidth for all positive loss coefficients. However we then go on to discuss the practical implications of this result and suggest that for any real-world scenario this result will not occur. We argue that truncated stable distributions are more realistic and we investigate the effective bandwidths of these.

3.1. The infinite moment generating function

Recall from Section 2 that the effective bandwidth of an arrival process, $A[i]$, can be determined from

$$\Lambda(\theta) = \lim_{n\to\infty} \frac{1}{n} \log E[\exp(\theta(A[1] + \cdots + A[n]))]. \qquad (5)$$

But $E[\exp(\theta(A[1] + \cdots + A[n]))]$ is merely an expression for the moment generating function of the arrival process which can also be expressed as

$$M(\theta) = \int_{-\infty}^{\infty} f(x).e^{\theta x} dx. \qquad (6)$$

$f(x)$ is the pdf of the arrival process. If this is stable then we know that $f(x)$ decays asymptotically in accordance with some power law,

$$\lim_{|x|\to\infty} f(x;\alpha,\beta) = C(\alpha,\beta)|x|^{-\alpha}. \qquad (7)$$

Therefore the integral in (6) consists of a term that grows exponentially in x (for all $\theta > 0$) and one that decays in accordance with a power law. If the asymptotic tail behaviour in (7) comes into effect over the ranges $x \le l$ and $x \ge u$ ($l \le u$) then

$$\begin{aligned} M(\theta;\alpha,\beta) \quad = \quad & [\int_{-\infty}^{l} C(\alpha,\beta)|x|^{-\alpha}.e^{\theta x} dx \\ & + \int_{u}^{\infty} C(\alpha,\beta)|x|^{-\alpha}.e^{\theta x} dx \\ & + \int_{l}^{u} f(x).e^{\theta x} dx]. \end{aligned} \qquad (8)$$

It is possible to show that the second term on the RHS of (8) is infinite for $\theta > 0$. We can therefore conclude that the effective bandwidth for all realistic values of the loss coefficient will be infinite.

3.2. The adapted moment generating function

At first sight the result in the previous section is disheartening since it suggests that if we attempt to estimate the effective bandwidth of any stable process then the result will be infinite. However if we consider a real world scenario we know that it is possible to model teletraffic with a stable model. In fact the result in the previous section is due to the fact that the moment generating function considers the entire probability space. In reality some form of truncation will occur because (i) negative arrivals are never permitted and (ii) the network will have some upper bound on its maximum transfer rate.

In order to investigate the growth of the moment generating function we need to rewrite (6) in a slightly different form

$$M(\theta, T;\alpha,\beta) = \int_{-T}^{T} e^{\theta x} f(x;\alpha,\beta) dx. \qquad (9)$$

The moment generating function has been limited by the value $T > 0$ (hence equation (6) can not be considered a true moment generating function, we will use the term adapted moment generating function to distinguish it from (6)) and we have included the stable parameters α and β. So by calculating the pdf of a stable distribution with characteristic exponent α and skew β we can find $M(\theta, T;\alpha,\beta)$ over a range of T.

In fact it is possible to obtain an exact expression for the adapted moment generating function in the cases where an expression for the pdf exists. In the case when $\alpha = 2$ an expression for the pdf is known and we can write

$$M(\theta, T, 2, 0) = e^{\frac{\theta^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-T-\theta}^{T-\theta} e^{\frac{x^2}{2}} dx. \qquad (10)$$

Using the fact that $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\frac{x^2}{2}} dx$ and $\Phi(-z) = 1 - \Phi(z)$ we can write

$$M(\theta, T, 2, 0) = e^{\frac{\theta^2}{2}} (2\Phi(T - \theta) - 1). \qquad (11)$$

We use the above expression in the next two subsections to compare with the approximate techniques for calculating the adapted moment generating function.

3.2.1. Fourier transform approximation to the pdf

The most obvious way to obtain an approximation of the pdf of a stable distribution is to take the inverse Fourier transform (IFT) of a sample of the characteristic function [7]. By suitably selecting the IFT size and the distance between the IFT points it is possible to generate a pdf approximation over any range of x. In Figure 2 the approximate pdf and adapted moment generating function are plotted for three values of α and $0 < T \leq 10$.

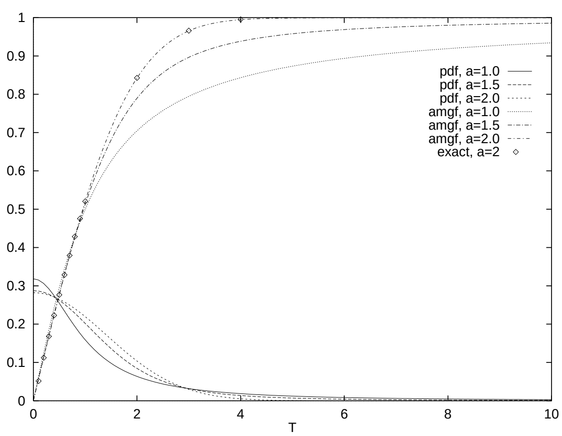

Figure 2: *The approximate pdf and adapted moment generating function for $\theta = 0.0001$ and α=1.0, 1.5 and 2.0 using the IFT. The exact values for the $\alpha = 2$ case are also given.*

3.2.2. Asymptotic expansion approximation to the pdf

Another pdf approximation technique is that of asymptotic expansion of the power series [7]. A SαS pdf can be approximated by

$$f(x, \alpha, 0) = \frac{1}{\pi x} \sum_{k=1}^{\infty} \frac{(-1)^k}{K!} \Gamma\left(\frac{2k+1}{\alpha}\right) x^{2k} \qquad (12)$$

for $1 \leq \alpha \leq 2$. Several extensions to this technique can be employed to increase the accuracy and reduce rounding errors. We approximated the pdf and adapted moment generating function for the same θ, α and T as for the technique in Section 3.2.1. A comparison between these results

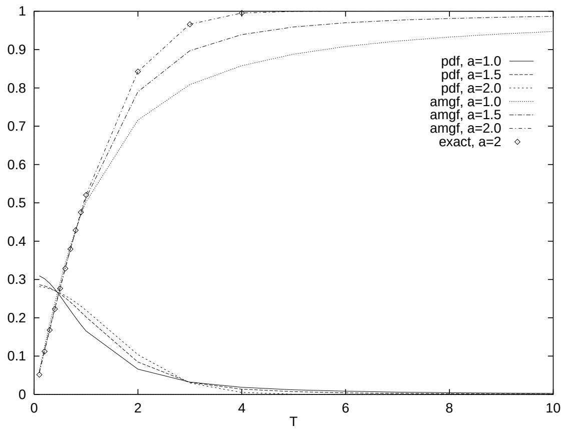

Figure 3: *The approximate pdf and adapted moment generating function for $\theta = 0.0001$ and α=1.0, 1.5 and 2.0 using asymptotic expansion. The exact values for the $\alpha = 2$ case is also given.*

and those in Section 3.2.1 are in very good agreement (to within 1%). It would seem reasonable to suggest that the differences in the results are due to inaccuracies in the approximation technique and that the estimates for the adjusted moment generating function are approximately correct.

4. SIMULATION RESULTS

Now consider the Norros model with the Gaussian innovations replaced by iid symmetric stable innovations,

$$A[i] = m + \sqrt{am} L_{\alpha,0,0.5}[i]. \qquad (13)$$

If the $L_{\alpha,0,0.5}[i]$ process is truncated to $T = 10$ and $m = 10$, $a = 1$ then a semi-definite positive arrival process with $A[i] \in (0, 20)$ is produced. We estimate $\Lambda(\theta)$ for this process using the technique in [4]. The result for $\theta = 0.1$ was 0.1018377, which suggests that the effective bandwidth is 10.18377. When we serviced the source at this rate we obtained the buffer occupancy probability curve in Figure 4.

This is a positive result in that it suggests that the on-line estimation techniques developed by Crosby *et al.* can still be applied to data with a stable innovation process. Now we wish to determine whether we can use (9) to estimate the effective bandwidth of the model. We applied the techniques described in Sections 3.2.1 and 3.2.2 to estimate the adjusted moment generating function with the parameters $\alpha = 1.5$, $\beta = 0$, $T = 10$ and $\theta = 0.1$. The results were

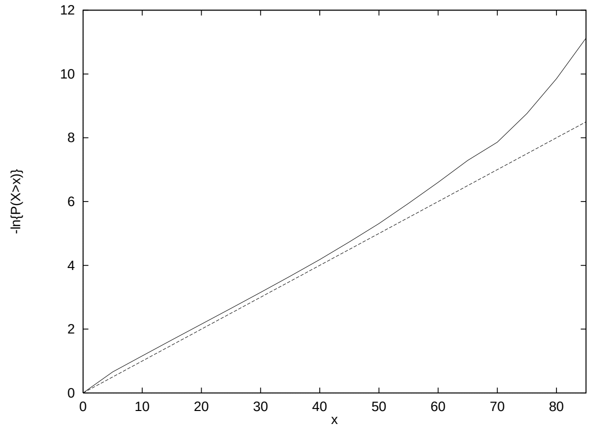

Figure 4: *The buffer occupancy curve for the truncated iid stable Norros model serviced at 10.1838 bits/s with the expected value from effective bandwidth theory (dashed line).*

1.005169 and 1.003800 respectively. The effective bandwidth can be estimated for the model using the following,

$$\delta(\theta) = 10 + \frac{1}{\theta} \log \mathbf{M}(\theta, T, \alpha, \beta). \tag{14}$$

So the effective bandwidth estimates are 10.0518 and 10.0379. These may seem close to the 10.18377 value but in fact the system is very sensitive to $\delta(\theta)$. This is obvious when we plot the buffer occupancy plot for a service rate of 10.0518 Figure 5

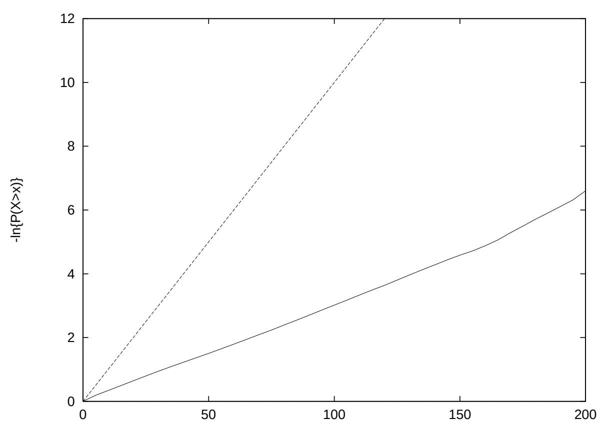

Figure 5: *The buffer occupancy curve for the truncated iid stable Norros model serviced at 10.0518 bits/s with the expected value from effective bandwidth theory (dashed line).*

So the adjusted moment generating function estimates of the effective bandwidth are less than the true value for the given loss figure.

5. CONCLUSIONS

The effective bandwidth of all stable distributions with $\alpha < 2$ was shown to be infinite so we investigated the truncated stable distribution. This was because such distributions are more realistic in real world scenarios. However the analytical effective bandwidth estimates based on this technique underestimated the required service rate. This is because of a mismatch between the adapted moment generating function and the true pdf of the truncated stable distribution. It may be possible to bias the adapted moment generating function to correct for this mismatch but this remains an area of future work. The online estimation technique worked well for the truncated stable model suggesting that effective bandwidths still have a possible role to play in CAC on ATM networks.

6. REFERENCES

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