THE GABOR EXPANSION BASED POSITIVE DISTRIBUTION

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Abstract *- This paper presents and studies a time frequency distribution obtained from a Gabor expansion of a signal. The distribution is named the Positive Gabor Spectrogram, and is a new positive-like distribution with correct marginal distributions. Two side effects of correct marginals are non-additivity and time frequency fading. These are phenomena of a statistical correct distribution which do not agree with our intuitive expectation of a time frequency representation.*

I. INTRODUCTION

The purpose of *Joint Time Frequency Analysis (JTFA)* is to map a one-dimensional function of time or frequency into a two-dimensional function of time and frequency. Two popular approaches are *Wigner-Ville (WV)* modified distributions and statistical correct positive distributions. In this paper, we will show how the latter type of distributions can be obtained from a Gabor expansion of the function.

The applicability of the Gabor expansion in JTFA has in several cases been shown to be advantageous. The *Gabor Spectrogram (GS)* is a very powerful distribution developed by Qian and Chen[1], and is obtained by combinding the WV distribution and the Gabor expansion. The problem with negative cross-terms in the WV can now be localized and reduced efficiently. By changing the number of Gabor coefficients, the spectral resolution of the GS can be changed in a more advantageous way than known from the *Fourier Spectrogram (FS)*. However, the result depends strongly on how the Gabor expansion is made.

The *Positive Gabor Spectrogram (PGS)* is a new positive distribution [2] with marginals corresponding to the *Instantaneous Power (IP)* and the *Power Spectrum (PS)*. It is obtained by combining the Gabor expansion and the IP/PS and is a statistical correct distribution. Different PGS can be obtained for the same function by changing the particular Gabor expansion. Which Gabor expansion to use will for the GS and PGS depend on the particular function and application. However, it is important that the Gabor expansion is an orthogonal-like Gabor expansion [3], because the resulting Gabor coefficients reflect well local time and frequency behaviour of the signal.

II. POSITIVE GABOR SPECTROGRAM

The *Positive Gabor Spectrogram (PGS)* is designed as a positive distribution with marginals corresponding to the IP and PS:

$$\begin{aligned} &C_x(t,\omega) \geq 0 \\ &\int C_x(t,\omega)\,dt = |X(\omega)|^2 \\ &\frac{1}{2\pi}\int C_x(t,\omega)\,d\omega = |x(t)|^2 \end{aligned} \tag{1}$$

This class of distributions is described by the Cohen-Posch class of distributions. The significance of this class is that there is not only one distribution which satisfies our demands, but an infinite number of distributions. The remaining problem is to find a meaningful one for a given signal. One of the main contributions to this problem has been given by Loughlin [4]. He used an information theoretic approach to estimate a distribution. This entropy optimization problem involved the principle of minimum cross-entropy, and he therefore named his distribution a *Minimum Cross-Entropy Time-Frquency Distribution (MCE-TFD)*. Because positive distributions are distributions in a statistical sense, Loughlin named this class of distributions, *proper distributions* which have been widely accepted. Now, instead of following Loughlin's approach to a positive distribution, we propose a similar distribution based on the Gabor theory.

An important task in constructing a meningful TF energy distribution, is to control the contributions from the inevitable *cross-terms (CT)*. These CT are not a phenomenon limited to bilinear distributions, but is due to the quadratic nature of an energy representation. As CT exist in all energy distributions, they are also present in 1D distributions like the IP and PS.

In order to construct a 2D distribution, it will be advantageous first to consider 1D distributions. This will reveal some of the nature of CT and enable us to construct a 2D positive energy distribution with correct marginals. The effect of CT is distinct when analyzing multicomponent signals, and in the following we will use a two component signal $x(t) = x_1(t) + x_2(t)$. The IP/PS are 1D energy distributions and are defined as the square of the modulus of the time and frequency representation of the signal, respectively:

$$
\begin{aligned}
|x(t)|^2 &= |x_1(t) + x_2(t)|^2 = |x_1(t)|^2 + |x_2(t)|^2 \\
&\quad + 2 \cdot Re\{x_1(t) x_2^*(t)\} \\
|X(\omega)|^2 &= |X_1(\omega) + X_2(\omega)|^2 = |X_1(\omega)|^2 + |X_2(\omega)|^2 \\
&\quad + 2 \cdot Re\{X_1(\omega) X_2^*(\omega)\}
\end{aligned}
\tag{2}
$$

It is clear that the energy representation of a two component signal is not the sum of the energy representation of the individual terms. This is sometimes noted by the fact that energy content is not additive (non-additivity). The third term is due to the quadratic nature, and is a CT, because it contains information about both components. Even though the CT can be negative, a 1D distribution cannot be negative by definition. The energy contribution of the CT is equal in both time and frequency domain, i.e.

$$\int Re\{x_1(t) x_2^*(t)\}\, dt = \frac{1}{2\pi}\int Re\{X_1(\omega) X_2^*(\omega)\}\, d\omega \tag{3}$$

From (3) it is concluded that the components have to be overlapping in both domains, before the CT contributes with energy! In the rest of this article, we use Gaussian functions that distinguish themselves by being easy to manipulate mathematically and are shape invariant under the Fourier transformation. To be specific, we use energy normalized functions:

$$
\begin{aligned}
g(t) &= (\pi\sigma^2)^{-0,25} \exp\left(-\frac{t^2}{2\sigma^2}\right) \\
G(\omega) &= (4\pi\sigma^2)^{0,25} \exp\left(-\frac{\omega^2\sigma^2}{2}\right)
\end{aligned}
\tag{4}
$$

and a particular TF shifted Gaussian function of the following form:

$$
\begin{aligned}
g_k(t) &= g(t - T_k)\, e^{jt\Omega_k} \\
G_k(\omega) &= G(\omega - \Omega_k)\, e^{-j(\omega - \Omega_k) T_k}
\end{aligned}
\tag{5}
$$

It is the nature of quadratic transformations to produce n^2 functions, if the input contains n functions. The sum of these n^2 functions will be positive by definition. In order to interpret the result, it would be desirable that the result would again consist of the same n number of <u>positive functions</u>. This would force the contributions from the $n^2 - n$ CT to be placed together with the auto-terms to form an n component function. This function will be called the *Positive Power Decomposition (PPD)*.

We will first find a possible PPD for two Gaussian functions. This PPD will then be extended to handle N Gaussian functions. This enables a PPD for arbitrary signals through the Gabor expansion!

However, before defining a PPD for functions, we will first show a simple example involving only numbers instead of functions. The absolute square of three arbitrary numbers is:

$$|1 + 4 - 2|^2 = \underbrace{|1|^2 + |4|^2 + |2|^2}_{\text{AT}} - \underbrace{12}_{\text{CT}} \tag{6}$$

The negative level of the CT will never exceed the positive level of the AT. Therefore it will be posible to divide the contribution from the CT between the components of the AT <u>and</u> preserve the positive level of each of the AT:

$$
\begin{aligned}
|1 + 4 - 2|^2 &= \left(|1|^2 + \frac{|1|^2}{|1|^2 + |4|^2 + |2|^2} \cdot (-12)\right) \\
&\quad + \left(|4|^2 + \frac{|4|^2}{|1|^2 + |4|^2 + |2|^2} \cdot (-12)\right) \\
&\quad + \left(|2|^2 + \frac{|2|^2}{|1|^2 + |4|^2 + |2|^2} \cdot (-12)\right) \\
&= 0,43 + 6,86 + 1,71
\end{aligned}
\tag{7}
$$

The PPD for two Gaussian functions consists of two functions, $P_1(t)$ and $P_2(t)$:

$$
\begin{aligned}
|x(t)|^2 &= |g_1(t)|^2 + |g_2(t)|^2 + 2 \cdot Re\{g_1^*(t) g_2(t)\} \\
&= P_1(t) + P_2(t)
\end{aligned}
\tag{8}
$$

It will always be possible to make this positive decomposition, because the negative part of the CT will never exceed the positive level of the AT. The constraint of the positive functions, $P_1(t)$ and $P_2(t)$ will be:

$$
\begin{aligned}
&(a) && P_1(t) = |g_1(t)|^2 + DCT_1(t) \geq 0 \\
&(b) && P_2(t) = |g_2(t)|^2 + DCT_2(t) \geq 0 \\
&(c) && \int DCT_1(t)\, dt = \frac{A_1^2}{A_2^2} \cdot \int DCT_2(t)\, dt \\
&(d) && DCT_1(t) + DCT_2(t) = 2 \cdot Re\{g_1^*(t) g_2(t)\}
\end{aligned}
\tag{9}
$$

Eqs. (9)**a**, **b** and **c** set the local and global constraints on the *decomposed cross-term* (DCT). The local constraint ensures positivity, and the global constraint divides the energy contribution of CT between the auto-terms according to the amplitude of the auto-terms. Notice that this constraint involves integration of part of a Gaussian function, which is very difficult. We will therefore differ from (9)**c**, and then later see the consequences of this choise. We now introduce the following:

$$DCT_1(t) = h(t) \cdot CT(t)$$
$$DCT_2(t) = (1 - h(t)) \cdot CT(t) \tag{10}$$

The limitations on $h(t)$ can be found from

$$|g_1(t)|^2 - h(t) \cdot CT(t) \geq 0$$
$$|g_2(t)|^2 - (1 - h(t)) \cdot CT(t) \geq 0$$
$$\Leftrightarrow \tag{11}$$
$$\frac{|g_1(t)|^2}{2A_1A_2 g(t - T_1) g(t - T_2)} \leq h(t) \leq 1 - \frac{|g_2(t)|^2}{2A_1A_2 g(t - T_1) g(t - T_2)}$$

A natural choice would be to divide the CT contribution according to the level of the auto-terms:

$$ht) = \frac{|g_1(t)|^2}{|g_1(t)|^2 + |g_2(t)|^2} \tag{12}$$

Observe that using (12) the positive constraint is met.

The example with two Gaussian functions can be extended to N*M Gaussian functions, and thereby a Gabor expansion of an arbitrary function or signal. The PPD will therefore be a collection of positive functions, and the sum will be the Instantaneous Power. This is done in the following way: First the signal is represented by the following Gabor expansion:

$$x(t) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} a_{mn} g(t - mT) e^{-jtm\Omega} \tag{13}$$

and the corresponding auto- and cross-terms are found to be

$$AT(t) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left| a_{mn} g(t - mT) e^{-jtm\Omega} \right|^2$$
$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} AT_{mn}(t) \tag{14}$$
$$CT(t) = |x(t)|^2 - AT(t)$$

The $AT(t)$ is the sum of $M \cdot N$ positive Gaussian functions $AT_{mn}(t)$. Then PPD can be found by dividing the sum of cross-term $CT(t)$ between the auto-terms $AT_{mn}(t)$. Following the idea of (7), the PPD will have the following form:

$$|x(t)|^2 = AT(t) + CT(t)$$
$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} p_{mn}(t) \tag{15}$$

where

$$p_{mn}(t) = \left| a_{mn} g(t - mT) e^{-jtm\Omega} \right|^2 \cdot \left(1 + \frac{CT(t)}{AT(t)} \right)$$

The same decomposition can be made for the PS. When the Gabor expansion is Fourier transformed, the PPD enables a division of the CT between the autoterms. The Fourier transformed signal is represented by the following Gabor expansion:

$$X(\omega) = FT\{x(t)\} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} a_{mn} G(\omega - n\Omega) e^{-j(\omega - n\Omega)mT} \tag{16}$$

The corresponding auto- and cross-terms can be found to be

$$AT(\omega) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left| a_{mn} G(\omega - n\Omega) e^{-j(\omega - n\Omega)mT} \right|^2 \tag{17}$$
$$CT(\omega) = |x(\omega)|^2 - AT(\omega)$$

The PPD can now be obtained by dividing the sum of cross-term $CT(t)$ between the auto-terms:

$$|X(\omega)|^2 = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} P_{mn}(\omega) \tag{18}$$

where

$$P_{mn}(\omega) = \left| a_{mn} G(\omega - n\Omega) e^{-j(\omega - n\Omega)mT} \right|^2 \cdot \left(1 + \frac{CT(\omega)}{AT(\omega)} \right)$$

The advantage of the PPD of the IP and the PS respectively, is now clear, because it enables the construction of a positive time-frequency distribution, which in the following is referred to as the *Positive Gabor Spectrogram (PGS)*:

$$PGS(t, \omega) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \frac{p_{mn}(t) \cdot P_{mn}(\omega)}{\|P_{mn}\|^2} \tag{19}$$

It is clear that the PGS is positive, as the PPD terms are positive. The time marginal is also preserved because:

$$\frac{1}{2\pi} \int PGS(t, \omega) \, d\omega = |x(t)|^2 \tag{20}$$

In the case of the frequency marginals, we have:

$$\int PGS(t, \omega) \, dt = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} P_{mn}(\omega) \frac{\int p_{mn}(t) \, dt}{\|P_{mn}\|^2} \tag{21}$$

Hence, we have a problem because we cannot ensure fulfilment of the global constraint which controls the division of the CT energy contribution. This means that $\|p_{mn}\|^2 \neq \|P_{mn}\|^2$ which is a main condition of (21). We can therefore not be sure that the frequency marginal is correct. but in practice, the problem is minor.

To test the PGS, an analysis is made of four test signals, and the results are placed in Figure 1. The results of the PGS are compared to the FS and WV, because these distributions have properties which we want the PGS to possess : positivity and correct marginals.

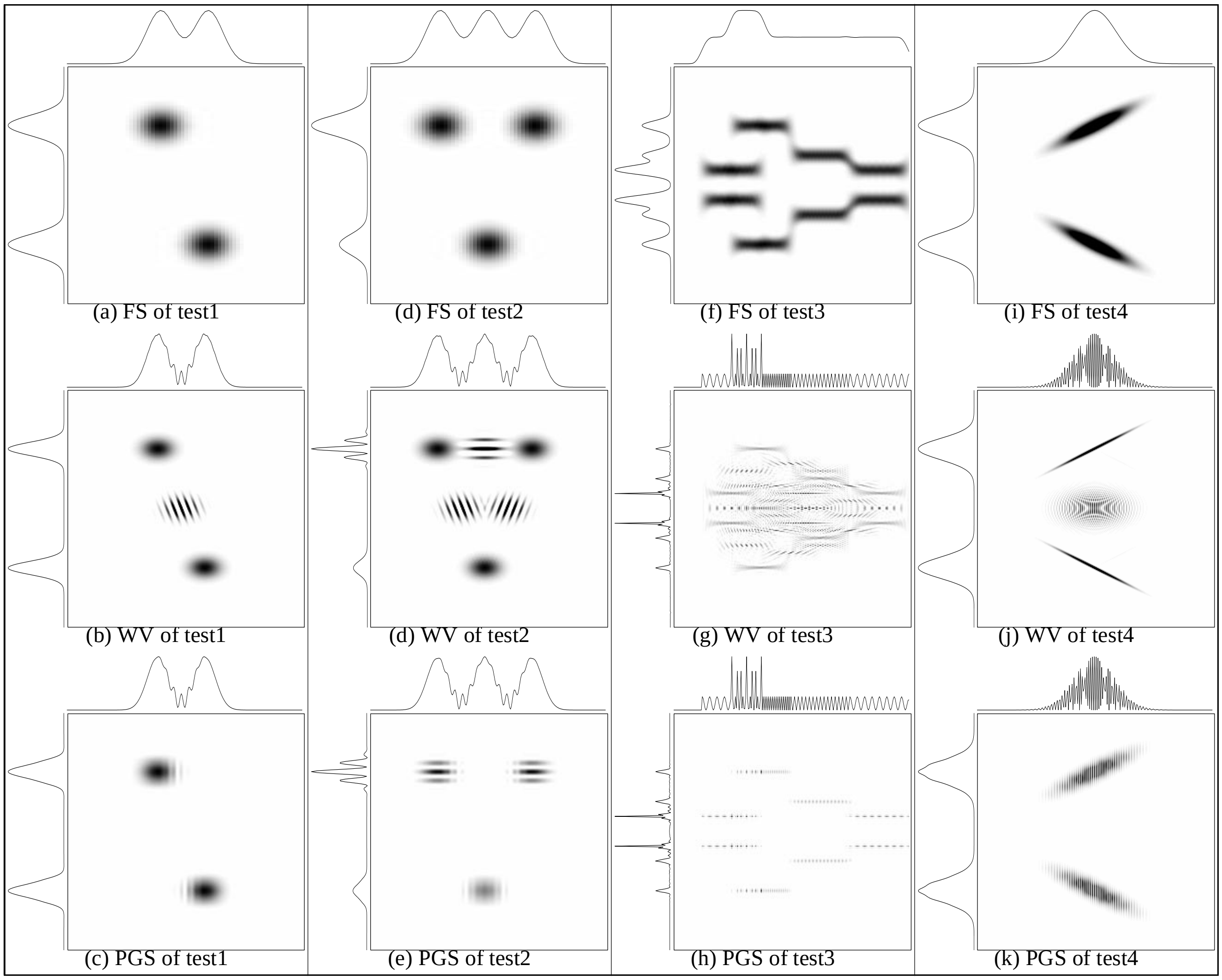

Figure 1. Result of the FS, the WV and the PGS for four test signals.

III. CONCLUSION

We have shown that positive distribution can be constructed using the Gabor expansion. However, the result have missing focus for signal component wich are not oriented in vertical or horizontal lines in the time-frequency plan [5]. The effect of correct marginals is TF fading, which means that every zero-crossing in the PS and IP is fading the distribuiton to zero. The resulting distribution of a real signal will therefore have a very spiky shape. When the distribution was implementated, the corresponding distribution was sampled in discrete points. However, when these points were closed or identical with the zero-crossing, the resulting distribution was misleading. An effect of the frequency marginal is that *The results of our analysis of measurements made "Monday" will be effected of our measurements made "Tuesday"* (non-additivity).

IV. REFERENCES

[1] S. Qian and D. Chen, "Decomposition of the Wigner-Ville Distribution and Time-Frequency Distribution Series", IEEE Trans. on Signal Processing. Vol. 42, No. 10, Oct. 1994, pp. 2836-2842.

[2] F. Pedersen, "Joint Time Frequency Analysis in Digital Signal Processing", Ph.D. Dissertation, May 1997, Aalborg University, Denmark, ISBN 87-986453-0-7

[3] S. Qian and D. Chen, "Discrete Gabor Transform", IEEE Trans on Signal Processing. Vol. 41, No. 7, July 1993, pp. 2429-2438.

[4] P. J. Loughlin, "Time-Frequency Energy Density Functions: Theory and Synthesis", Ph.D. Dissertation, Univ. of Washington, 1992.

[5] J. R. Fonollosa, " Positive Time-Frequency Distributions based on Joint Marginal Constraints", IEEE Trans. on SP, Vol. 44, No. 8, August 1996, pp. 2086-2091.