ANISOTROPIC DIFFUSION AND LOCAL MONOTONICITY

Scott T. Acton
School of Electrical and Computer Engineering
Oklahoma State University
Stillwater, Oklahoma 74078
sacton@okstate.edu

ABSTRACT

This paper investigates the relationship between anisotropic diffusion and local monotonicity. A diffusion technique that has locally monotonic root signals is presented. The enhancement algorithm rapidly converges to a locally monotonic signal of the desired degree. It is shown that the diffusion coefficient used here is the only formation that guarantees idempotence for locally monotonic signals. The signals resulting from locally monotonic diffusion are closer to the original signals than the corresponding median root signals. Furthermore, the diffusion algorithm does not have a difficulty with alternating signals, as does the median filter. In contrast to other anisotropic diffusion techniques, the diffusion method given here does not preserve outliers and does not require a gradient magnitude threshold in the diffusion coefficient.

1. INTRODUCTION

The success of a signal enhancement technique depends upon the metric used to evaluate signal smoothness. Qualitatively, an enhancement process is desired that eradicates noise while preserving information-rich signal transitions – the edges. For discrete signals, the traditional notion of evaluating smoothness by continuity does not apply. Moreover, limiting the instantaneous rate of change in a signal leads to destruction of the signal edges.

The smoothness of a discrete signal may be assessed by computing its degree of *local monotonicity*. Locally monotonic signals are nonincreasing or nondecreasing within all contiguous subsequences of specified lengths. Therefore, local monotonicity limits the oscillations in the signals without constraining the magnitude of signal transitions. Specifically, a length-N signal $\mathbf{I}$ is locally monotonic of degree d (or LOMO-d) if each contiguous subsequence of length d (e.g., $\{I(x), I(x+1), \ldots, I(x+d-1)\}$ is monotonic. Every signal is LOMO-2, and LOMO-N is the highest degree possible. The *lomotonicity* of a signal is the highest degree of local monotonicity maintained by the signal. Of course, for $d > e$, a LOMO-d signal is also LOMO-e.

The power of local monotonicity was first discovered in the analysis of root signals produced by the median filter [6]. Since that time, Restrepo and Bovik [4] and Sidiropoulos [5] have developed methods to solve the LOMO regression problem. In this framework, the computation of a LOMO signal that resembles the original (and possibly noisy) signal is treated as an optimization problem. This paper investigates the creation of LOMO signals via a simple adaptive diffusion mechanism.

2. DIFFUSION

Diffusion processes implemented by partial differential equations are useful for enhancing signals and producing a family of signal descriptions that form a scale-space. Anisotropic diffusion algorithms are distinguished by the ability to avoid diffusion (and thus smoothing) across signal edges [3]. The rate of diffusion is controlled by a diffusion coefficient, which is typically a decreasing function of gradient magnitude. For continuous-domain signals, the diffusion process may be modeled by

$$\frac{\partial I(\mathbf{x})}{\partial t} = \operatorname{div}\left[c(\mathbf{x})\nabla I(\mathbf{x})\right] \qquad (1)$$

where ∇ is the gradient operator, div is the divergence operator (div $x = \nabla \bullet x$), $c(\mathbf{x})$ is the diffusion coefficient at location $\mathbf{x}$ and $I(\mathbf{x})$ is the signal intensity. For discrete-domain signals, the PDE of (1) may be discretized by the following Jacobi iterate:

$$\left[I(\mathbf{x})\right]_{t+1} \leftarrow \left[I(\mathbf{x}) + \Delta T \sum_{p=1}^{\Omega} c_p(\mathbf{x}) \nabla I_p(\mathbf{x})\right]_t \qquad (2)$$

where ΔT is the time step, t represents iterations, p enumerates the diffusion paths (directions), and Ω is the number of diffusion paths. For 1-D signals, we can utilize a simplified expression:

$$\left[I(x)\right]_{t+1} \leftarrow \left\{I(x) + (1/2)\left[c_e(x)\nabla I_e(x) + c_w(x)\nabla I_w(x)\right]\right\}_t \qquad (3)$$

where $\nabla I_e(x)$ and $\nabla I_w(x)$ are differences with respect to the "eastern" and "western" neighbors, defined by

$$\nabla I_e(x) = I(x + h_e) - I(x) \qquad (4)$$

and

$$\nabla I_w(x) = I(x - h_w) - I(x)\,. \qquad (5)$$

h_e and h_w are the sample spacings used to estimate the directional derivatives in the eastern and western directions, respectively.

A number of diffusion coefficients have been explored in the literature. Traditionally, the coefficients take the form of [3]

$$c_p(x) = \exp\left\{-\left[\frac{\nabla I_p(x)}{k}\right]^2\right\} \qquad (6)$$

where k is an gradient magnitude threshold and determines which edges will be retained in the diffusion process. The parameter k is difficult to define analytically for general application. In addition, diffusion coefficients of the form of (6) preserve outliers due to noise where the outliers have large gradient magnitudes. To correct this limitation, new diffusion coefficients have been proposed [1] that use a pre-smoothed image to estimate the gradient magnitudes. This approach, however, introduces a linear diffusion process into the nonlinear diffusion process, limiting edge retention and localization.

Diffusion coefficients have been designed that allow the diffusion operation to converge to a constant or piecewise constant signal [7]. An anisotropic diffusion algorithm that converges to LOMO signals has not yet been explored.

3. LOCALLY MONOTONIC DIFFUSION

Consider the following diffusion coefficient function:

$$c_p(x) = \frac{1}{\left|\nabla I_p(x)\right|}. \quad (7)$$

Given the restriction that the diffusion coefficient be a smooth and nonincreasing function of gradient magnitude, we must modify (7) for the cases $\nabla I_w(x) = 0$ and $\nabla I_e(x) = 0$. We set $\nabla I_w(x) \leftarrow \nabla I_e(x)$ in the case of $\nabla I_w(x) = 0$, and $\nabla I_e(x) \leftarrow \nabla I_w(x)$ when $\nabla I_e(x) = 0$. The case where both differences are zero does not affect the diffusion operation. If (7) is substituted into (3), we have

$$[I(x)]_{t+1} \leftarrow \left(I(x) + (1/2)\{\mathrm{sgn}[\nabla I_e(x)] + \mathrm{sgn}[\nabla I_w(x)]\}\right)_t. \quad (8)$$

This simple iterate implements diffusion to a locally monotonic signal – LOMO diffusion. The lomotonicity of the root signal depends on the sample spacing used to estimate the gradient magnitude values. For example, if $h_e = 1$ in (4) and $h_w = 1$ in (5), (8) converges to a LOMO-3 signal. If $h_e = 1$ and $h_w = 2$, a LOMO-4 diffusion algorithm is produced. Higher degrees of lomotonicity may be achieved by the use of multiple passes with different sample spacings.

For input signal **I**, let the signal that results from iterating (8) to a root signal be denoted by ld(**I**, h_w, h_e). Then let $\mathrm{ld}_d(\mathbf{I})$ denote a diffusion that results in a LOMO-*d* (or greater) signal. Then the lomo-3 diffusion can be given by:

$$\mathrm{ld}_3(\mathbf{I}) = \mathrm{ld}(\mathbf{I},1,1). \quad (9)$$

LOMO-4 signals are produced by

$$\mathrm{ld}_4(\mathbf{I}) = \mathrm{ld}(\mathbf{I},1,2). \quad (10)$$

Higher degree LOMO signals are computed using

$$\mathrm{ld}_5(\mathbf{I}) = \mathrm{ld}(\mathrm{ld}(\mathbf{I},2,2),1,2), \quad (11)$$

$$\mathrm{ld}_6(\mathbf{I}) = \mathrm{ld}(\mathrm{ld}(\mathbf{I},2,3),1,2), \quad (12)$$

$$\mathrm{ld}_7(\mathbf{I}) = \mathrm{ld}(\mathrm{ld}(\mathrm{ld}(\mathbf{I},3,3),2,3),1,2), \quad (13)$$

$$\mathrm{ld}_8(\mathbf{I}) = \mathrm{ld}(\mathrm{ld}(\mathrm{ld}(\mathbf{I},3,4),2,3),1,2), \quad (14)$$

$$\mathrm{ld}_9(\mathbf{I}) = \mathrm{ld}(\mathrm{ld}(\mathrm{ld}(\mathrm{ld}(\mathbf{I},4,4),3,4),2,3),1,2), \quad (15)$$

$$\mathrm{ld}_{10}(\mathbf{I}) = \mathrm{ld}(\mathrm{ld}(\mathrm{ld}(\mathrm{ld}(\mathbf{I},4,5),3,4),2,3),1,2), \quad (16)$$

and so on. In general, when iterating toward a LOMO-(2*m*+1) signal, the first LOMO diffusion is ld(**I**, *m*, *m*). For a LOMO-(2*m*+2) signal, the first operation is ld(**I**, *m*, *m*+1). The subsequent diffusion is ld(**I**, *m*-1, *m*) and then ld(**I**, *m*-2, *m*-1). This progression continues until ld(**I**, 1, 2) is implemented.

4. ANALYSIS

Theorem 1: For 1-D anisotropic diffusion using (3), diffusion coefficients of the form $c_d(x) = \frac{\kappa}{\left|\nabla I_d(x)\right|}$ uniquely allow a diffusion operation that is idempotent for LOMO-*d* $(d \geq 3)$ signals.

Proof: The lomotonicity of a signal may be evaluated from the sign skeleton of its difference signal alone. If the length-3 segment centered at location *x* in **I** is not monotonic, then $\mathrm{sgn}[\nabla I_w(x)] = \mathrm{sgn}[\nabla I_e(x)]$, and *I*(*x*) is said to be a non-LOMO point and must be changed. Otherwise, *I*(*x*) must remain unchanged (for idempotence on LOMO-3 signals). Assume $\mathrm{sgn}[\nabla I_w(x)] \neq \mathrm{sgn}[\nabla I_e(x)]$ and consider two cases: $\nabla I_w(x) = -\nabla I_e(x) + \varepsilon$ and $\nabla I_w(x) = -\nabla I_e(x) - \varepsilon$ (where $\varepsilon > 0$). Using (3), we assert that:

$$c_e(x)\nabla I_e(x) = c_w(x)\nabla I_w(x) \quad (17)$$

for *I*(*x*) to remain unchanged. Combining the two cases, we have

$$\begin{aligned} &c(\left|\nabla I_w(x) - \varepsilon\right|)[-\nabla I_w(x) + \varepsilon] \\ &= c(\left|\nabla I_w(x) + \varepsilon\right|)[-\nabla I_w(x) - \varepsilon] \end{aligned} \quad (18)$$

where $c(|g|)$ is the diffusion coefficient for a gradient of *g*. Under the stated assumptions for the diffusion coefficient, the only solutions to (18) are diffusion coefficients of the form $c_p(x) = \frac{\kappa}{\left|\nabla I_p(x)\right|}$. So, only diffusion using (7) will leave the length-3 (or greater) monotonic subsequences unchanged. Proof that (7) will always change the non-LOMO points can be given by examination of (8) when $\mathrm{sgn}[\nabla I_w(x)] = \mathrm{sgn}[\nabla I_e(x)]$. So, diffusion using (7) is idempotent for LOMO-3 signals. It is also idempotent for LOMO-*d* signals, where $d > 3$, since every signal that is LOMO-*a* is also LOMO-*b* if $a \geq b$.

Corollary: $\mathrm{ld}(\mathbf{I},1,1)$ will converge to a LOMO-3 signal in a finite number of iterations that is bounded above by the absolute value of the largest difference between two neighboring samples.

Proof: The only non-LOMO points possible are positive-going and negative-going outliers. By inspection of (8) and use of Theorem 1, each positive-going outlier will be reduced by a value of 1 until it is equal to one of its neighbors. Negative-going outliers will be incremented with each diffusion iteration. So, the solution will converge to a LOMO-3 signal. Let

$$T(x) = \min[\nabla I_w(x), \nabla I_e(x)]\,1\{\mathrm{sgn}[\nabla I_w(x)] = \mathrm{sgn}[\nabla I_e(x)]\} \quad (19)$$

where $1(\cdot)$ is the indicator function. The maximum number of iterations needed for convergence to a LOMO root signal is

$$\tau = \max[T(x) : 0 \leq x \leq N-1]. \quad (20)$$

As mentioned, the median filter produces LOMO root signals. The results from the literature can be summarized by Theorem 2:

Theorem 2 [6][2]: The output of a length $w = 2m+1$ median filter median(**I**) equals **I** if and only if **I** is LOMO-(m+2). Suppose that the 1-D signal **I** contains at least one monotonic segment of length $m+1$. Then the $w = 2m + 1$ median filter will reduce a length-N signal to a root signal that is LOMO-(m+2) in at most $(N - 2)/2$ repeated passes.

From Theorem 2, we can note that the median filter will require a significant number of iterations to reach the root signal on long signals. Also, a restriction is placed on the initial signal – it must contain a monotonic subsequence. With LOMO diffusion, the convergence to a root is not dependent on the length of the signal, and LOMO diffusion will produce a LOMO signal of the desired degree regardless of the input signal (see Fig. 1). Unlike traditional diffusion algorithms, LOMO diffusion does not retain outliers due to noise (see Fig. 2).

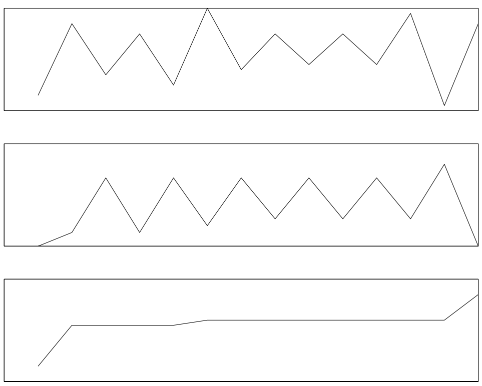

Fig. 1: Top: original alternating signal; Middle: w=3 median filter result after 31 iterations (no root signal possible); Bottom: LOMO-3 diffusion result after 11 iterations.

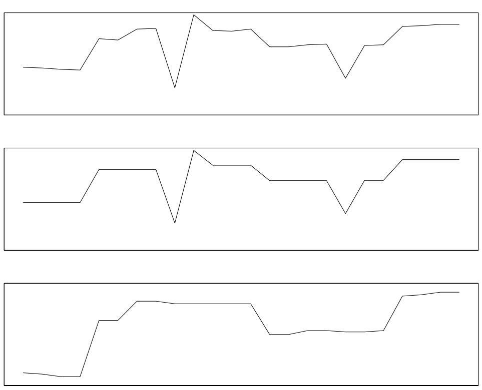

Fig. 2: Top: noisy input signal; Middle: diffusion result using (6); Bottom: LOMO-3 diffusion result (both use 85 iterations).

5. RESULTS AND CONCLUSIONS

To demonstrate the efficacy of LOMO diffusion, we generated LOMO root signals using the median filter and LOMO diffusion from 50 signals that were corrupted by Laplacian-distributed additive noise (SNR=10dB). Tables I and II summarize the results, and example output signals are shown in Figs. 3 and 4. One general conclusion is that LOMO diffusion provides a signal of higher lomotonicity and lower mean absolute error (MAE), compared to the median root. The LOMO diffusion algorithm is also quite efficient compared to the multiple iterations of the large (high w) median filters needed for producing root signals. Note that the signals of increasing lomotonicity form a scale-space that varies from fine to coarse. Also compare the LOMO diffusion of Fig. 4 to the noisy results of diffusion (using the same number of iterations) with the diffusion coefficient (6) in Fig. 5.

Table I: Results from LOMO-d diffusion on 50 signals of length N=64, corrupted with Laplacian-distributed noise.

Lomotonicity (d)	Avg. Lomotonicity	Avg. MAE	Avg. Iterations
3	3.00	4.27	41.52
4	4.16	6.44	48.76
5	5.40	7.60	77.24
6	6.98	8.63	82.22
7	10.92	9.91	111.54
8	15.18	10.63	114.32
9	21.90	11.80	145.66
10	26.18	12.31	133.96

Table II: Results from computing median root signals on 50 signals of length N=64, corrupted with Laplacian-distributed noise (d=m+2 in Theorem 2).

Lomotonicity (d)	Avg. Lomotonicity	Avg. MAE	Iterations (N-2)/2	Width (w)
3	3.00	4.98	31	3
4	4.22	7.40	31	5
5	5.38	8.60	31	7
6	6.72	9.47	31	9
7	7.60	10.44	31	11
8	9.62	11.26	31	13
9	12.78	11.90	31	15
10	13.62	14.05	31	17

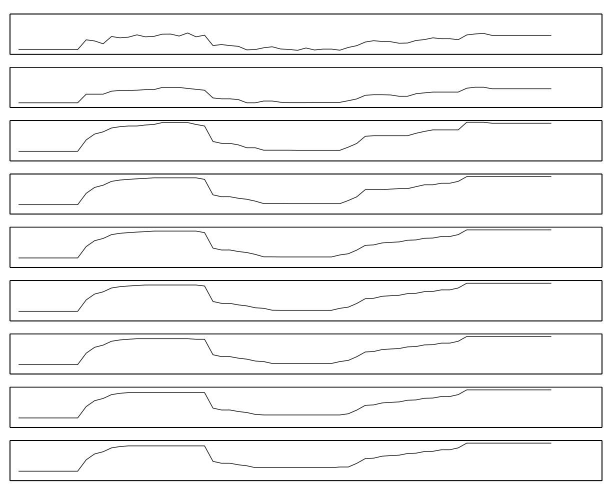

Fig. 3: From Top to Bottom: Noisy input signal and median filter roots for lomotonicity d=3 to d=10.

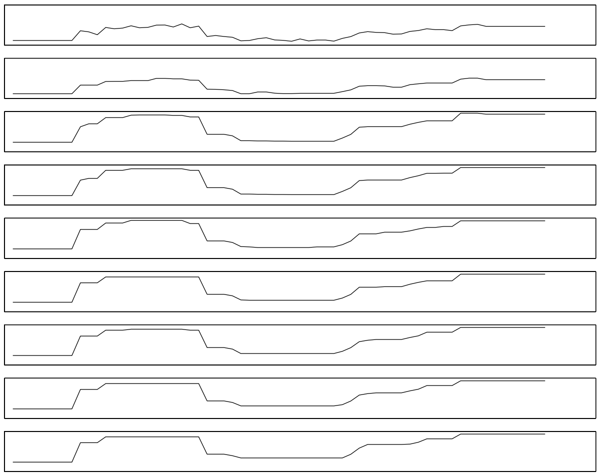

Fig. 4: From Top to Bottom: Noisy input signal and LOMO diffusion results for lomotonicity *d*=3 to *d*=10.

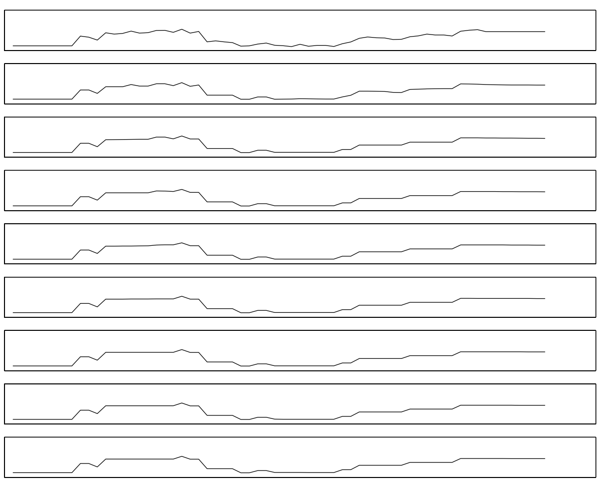

Fig. 5: From Top to Bottom: Noisy input signal and anisotropic diffusion results using (6). Each signal is produced using the same number of diffusion iterations used in corresponding signals in the Fig. 4 example.

A diffusion that converges to locally monotonic signals of the desired degree is presented. Analysis of the 1-D convergence properties is given and the imdempotence for LOMO signals is discussed. The results show multi-scale signal enhancement that preserves edges and removes impulses without additional filtering. LOMO diffusion does not utilize an *ad hoc* threshold nor does its convergence time depend on signal length. In future work, the extension of this powerful diffusion mechanism to 2-D signals will be analyzed. An example of LOMO diffusion on a noisy image (Fig. 6) is given in Fig. 7.

6. REFERENCES

[1] F. Catte, P.-L. Lions, J.-M. Morel, and T. Coll, "Image selective smoothing and edge detection by nonlinear diffusion," *SIAM J. Numer. Anal.*, vol. 29, pp. 182-193, 1992.

[2] N.C. Gallagher and G.L. Wise, "A theoretical analysis of the properties of median filters," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. ASSP-29, pp. 1136-1141, Dec. 1981.

[3] P. Perona and J. Malik, "Scale-space and edge detection using anisotropic diffusion," *IEEE Trans. on Pattern Anal. and Mach. Intell.*, vol. PAMI-12, pp. 629-639, 1990.

[4] A. Restrepo (Palacios) and A.C. Bovik, "Locally monotonic regression," *IEEE Trans. Signal Process.*, vol. 41, pp. 2796-2810, 1993.

[5] N. Sidiropoulos, "Fast digital locally monotonic regression," *IEEE Trans. Sig. Proc.*, vol. 45, pp. 389-395, 1997.

[6] S.G. Tyan, "Median filtering: Deterministic properties," in *Two-dimensional Signal Processing: Transforms and Median Filters*, T.S. Huang, ed. New York: Springer-Verlag, 1981.

[7] Y.-L. You, W. Xu, A. Tannenbaum and M. Kaveh, "Behavioral analysis of anisotropic diffusion in image processing," *IEEE Transactions on Image Processing*, vol. 5, pp. 1539-1553, 1996.

Fig. 6: Noisy 64x64 subimage.

Fig. 7: LOMO-3 diffusion result.