ALGORITHM FOR DECOMPOSING AN ANALYTIC SIGNAL INTO AM AND POSITIVE FM COMPONENTS

Ramdas Kumaresan *Ashwin Rao*
Department of Electrical Engineering
University of Rhode Island
Kingston, RI 02881.

ABSTRACT

An analytic signal permits unambiguous characterization of the phase and envelope of a real signal. But the analytic signal's phase-derivative *i.e.* the instantaneous frequency (IF) is typically a wild function and can take on values ranging fron negative infinity to positive infinity. Fortunately, any analytic signal can be decomposed into a minimum phase (MinP) signal component and an all-phase (AllP) signal component. While the MinP signal's log-envelope and its phase form a Hilbert transform pair, the AllP signal has a positive definite instantaneous frequency (PIF) unlike that of the original analytic signal. We propose an elegant computational algorithm that separates the MinP and AllP components of the analytic signal. The envelope of the MinP component corresponds to the AM and the PIF of the AllP component corresponds to the positive FM.

1. INTRODUCTION

The fundamental issues related to analytic signals were addressed by Gabor in 1946 [1], followed by Ville and others [2–4]. A signal is said to be analytic if its Fourier transform vanishes for either positive or negative frequencies. Such a representation permits an unambiguous characterization of a signal by its envelope and phase/frequency modulations. The phase-derivative (or IF) of an analytic signal has been extensively studied [2, 5]. A comprehensive review of IF of monocomponent signals (defined by Cohen [2]) has been provided by Boashash [5], along with discussions on existing algorithms and applications. The general impression among researchers has been that IF is meaningful only for narrowband or monocomponent signals [2]. In 1966, Voelcker studied the IF of analytic signals in the context of unifying various modulation methods. His studies were based on a non-linear representation of signals as product of elementary signals rather than sum of sinusoidal signals as in traditional Fourier analysis[6]. For a periodic bandlimited signal, the product-expansion simply means representing the signal by factoring the periodic signal's Fourier series. Non-periodic band-limited signals can be similarly treated as products of elementary signals (Cauchy-Hadamard product) but require difficult mathematics associated with the so-called entire functions [6]. However even when signals are not periodic and non-stationary, in many practical applications it is reasonable to work with a short segment of the signal and consider periodic extensions of it, as is common in short-time spectral analysis. In this paper we model a T second segment of a signal by a product representation model and decompose it into minimum-phase (MinP) and an all-phase (AllP) component. The MinP signal's envelope gives the AM component and the AllP signal is a pure phase signal whose IF or the FM is positive. An algorithm for such a decomposition is outlined in section 5. Recently, Poletti [7] has also modeled signals as products of elementary signals á la Voelcker. But our algorithm for separation of MinP and AllP components is believed to be novel.

This work was supported by the Air Force Contract AFOSR F49620-93-I-0026 .

2. PRODUCT REPRESENTATION OF SIGNALS

Let $s(t)$ be a periodic signal, with period T, consisting of $M+1$ complex sinewaves. Let $\Omega = 2\pi/T$ denote its fundamental angular frequency. Then

$$s(t) = \sum_{k=0}^{M} a_k e^{jk\Omega t} \quad , \tag{1}$$

where a_k's are the complex amplitudes of the sinusoids; $a_0 \neq 0$ and $a_M \neq 0$. We may regard $s(t)$ as a polynomial of degree M in the complex variable $e^{j\Omega t}$. Also, we may factor this polynomial into its M factors and rewrite $s(t)$ as

$$s(t) = \underbrace{\prod_{i=1}^{P}(1 - p_i e^{j\Omega t})}_{s_{\mathrm{MinP}}(t)} \underbrace{A_0 e^{j\omega_0 t} \prod_{i=1}^{Q}(1 - \frac{1}{q_i} e^{-j\Omega t})}_{s_{\mathrm{TMaxP}}(t)} \quad . \tag{2}$$

$p_1, p_2, .., p_P$, and $q_1, q_2, .., q_Q$ denote the polynomial's roots; $p_i = |p_i|e^{j\theta_i}$, $q_i = |q_i|e^{j\phi_i}$, $\omega_0 = Q\Omega$ and $A_0 = a_0(\prod_{i=1}^{Q} -q_i)$. p_is denote roots inside the unit circle in the complex plane, q_is are on or outside the unit circle. The subscript 'TMaxP' indicates that $s_{\mathrm{TMaxP}}(t)$ is a maximum phase (MaxP) signal that has been translated in frequency by ω_0.

The above expressions, representing a bandlimited periodic signal may be recognized as the counterpart of finite impulse response (FIR) filters in discrete-time systems theory [8]. More generally, if $s(t)$ consists of an infinite number of spectral lines, i.e., $S(\omega) = \sum_{k=0}^{k=\infty} a_k \delta(\omega - k\Omega)$, then analogous to an (infinite impulse response) IIR filter's system function, we can represent $s(t)$ over T secs to desired accuracy using sufficient number of poles and zeros as

$$s(t) = A_0 \underbrace{\frac{\prod_{i=1}^{P}(1-p_i e^{j\Omega t})}{\prod_{i=1}^{U}(1-u_i e^{j\Omega t})}}_{s_{\rm MinP}(t)} \quad \underbrace{e^{j\omega_0 t}\prod_{i=1}^{Q}(1-\frac{1}{q_i}e^{-j\Omega t})}_{s_{\rm TMaxP}(t)} \quad ; \quad (3)$$

p_is and q_is correspond to zeros inside and outside the unit circle respectively. u_is correspond to the signal's poles.

3. ENVELOPE AND PHASE RELATIONSHIPS

An elementary signal [6], $e(t)$, is defined as

$$e(t) = 1 - pe^{j\Omega t} \quad , \qquad (4)$$

where $p = |p|e^{j\theta}$. If $|p| < 1$ then $e(t)$ is called a MinP signal since no other signal with the same envelope has a smaller phase angle. Observe that $|e(t)| > 0$. Taking the natural logarithm of both sides and using the series expansion, $\ln(1-y) = \sum_{k=1}^{\infty}\frac{-y^k}{k}$, we get

$$\ln(1-pe^{j\Omega t}) = \sum_{k=1}^{\infty}\frac{-p^k e^{-jk\Omega t}}{k} \quad . \qquad (5)$$

After exponentiating both sides we get the following identity:

$$1 - pe^{j\Omega t} = \exp\{\sum_{k=1}^{\infty}\frac{-|p|^k}{k}\cos(k\Omega t + k\theta) + j\sum_{k=1}^{\infty}\frac{-|p|^k}{k}\sin(k\Omega t + k\theta)\} \quad . \qquad (6)$$

From the above expression we note that for an elementary MinP signal, $e(t)$, the logarithm of its envelope and its phase angle are related through the Hilbert transform. Similarly, for an elementary MaxP signal ($1 - qe^{j\Omega t}$, where $q = |q|e^{j\phi}$ and $|q| > 1$) we get the identity

$$1 - qe^{j\Omega t} = -qe^{j\Omega t}\exp\{\sum_{k=1}^{\infty}\frac{-|1/q|^k}{k}\cos(k\Omega t + k\phi) - j\sum_{k=1}^{\infty}\frac{-|1/q|^k}{k}\sin(k\Omega t + k\phi)\} \quad . \qquad (7)$$

The key difference between Eq.(8) and Eq.(7) is a sign change in their phase functions.

Using the above identities in Eq. 2 we have

$$s_{\rm MinP}(t) = e^{\alpha(t)+j\hat{\alpha}(t)} \qquad \text{and} \qquad (8)$$

$$s_{\rm TMaxP}(t) = A_0 e^{\beta(t)+j\left(\omega_0 t - \hat{\beta}(t)\right)} \quad , \quad \text{where} \qquad (9)$$

$$\alpha(t) = \sum_{k=1}^{\infty}\sum_{i=1}^{P} -\frac{|p_i|^k}{k}\cos(k\Omega t + k\theta_i) \quad \text{and} \qquad (10)$$

$$\beta(t) = \sum_{k=1}^{\infty}\sum_{i=1}^{Q} -\frac{1/|q_i|^k}{k}\cos(k\Omega t + k\phi_i) \quad . \qquad (11)$$

Thus $s(t)$ can be compactly represented as

$$s(t) = A_c e^{j\omega_c t} e^{\alpha(t)+j\hat{\alpha}(t)} e^{\beta(t)-j\hat{\beta}(t)} \quad , \qquad (12)$$

where A_c corresponds to the overall amplitude of the signal and ω_c denotes its 'carrier' frequency. ω_c is equal to $\omega_0 = Q\Omega$ plus any arbitrary frequency translation that the signal $s(t)$ may have been subjected to. The log-envelope, $\alpha(t) + \beta(t) + \ln A_c$ and phase, $\omega_0 t + \hat{\alpha}(t) - \hat{\beta}(t)$, of $s(t)$, based on the above expressions, are clearly not band-limited functions. It can be shown that $|s(t)|^2$ and $\frac{d\angle s(t)}{dt}|s(t)|^2$ are band-limited. The IF of $s(t)$ is the derivative of the phase of $s(t)$ and is simply $\omega_c + \dot{\hat{\alpha}}(t) - \dot{\hat{\beta}}(t)$, (where the dot stands for the first derivative) *i.e.* it consists of a d.c (corresponding to carrier frequency) and a sum of IFs of $s(t)$'s MinP and MaxP components. Clearly the behavior of the IF depends on the pole/zero locations of the signal $s(t)$ (for details see [9]). It is well-known that an analytic signal's IF could very well be negative. We now define an AllP signal whose IF is strictly positive and show that an arbitrary signal $s(t)$ observed over T seconds can be decomposed into a MinP signal and an AllP signal.

4. PERIODIC SIGNALS WITH POSITIVE INSTANTANEOUS FREQUENCY

Consider a signal, $z(t)$, which is a ratio of two elementary signals as follows:

$$z(t) = \frac{1 - qe^{j\Omega t}}{1 - \frac{1}{q^*}e^{j\Omega t}} \quad ; \qquad (13)$$

'$*$' denotes complex conjugation, $q = |q|e^{j\phi}$, and $|q| > 1$. Clearly, the above expression resembles the system function of an all-pass filter [8]. Simplifying the above expression, we find that $|z(t)|$ is a constant (equal to $|q|$) for all time and hence they are called All-Phase (AllP) signals. Plugging the expressions corresponding to the identities (Eq.7 and Eq.8) in Eq.(13) and taking the derivative of $z(t)$'s phase angle we get

$$\frac{d\angle z(t)}{dt} = \Omega\left(1 + 2\sum_{k=1}^{\infty}|1/q_i|^k\cos(k\Omega t + k\phi)\right) \quad . \qquad (14)$$

Since the right side of the expression in Eq. 14 is $\Omega(1 - |1/q|^2)|1 - 1/q^* e^{j\Omega t}|^{-2}$ and is analogous to a 'power spectrum' $z(t)$'s IF is always positive. We may generalize this result to the case of a signal consisting of a product of rational signals. If $z(t)$ is of the form

$$z(t) = \prod_{i=1}^{L}\frac{1 - q_i e^{j\Omega t}}{1 - \frac{1}{q_i^*}e^{j\Omega t}} \quad . \qquad (15)$$

then the phase angle contribution due to each of the L terms in the above equation adds up, and the corresponding IF is

$$\frac{d\angle z(t)}{dt} = \Omega\sum_{i=1}^{L}\left(1 + 2\sum_{k=1}^{\infty}|1/q_i|^k\cos(k\Omega t + k\phi_i)\right) \quad . \qquad (16)$$

Since each of the L terms in the above summation is positive, the IF of the entire signal $z(t)$ is positive. These results are analogous to well known results in discrete time all-pass systems, where the equivalent of IF is the group delay; our derivation is slightly different than the one given in Oppenheim and Schafer (page 238) [8].

Now consider $s(t)$ given by Eq. 2. It can be alternatively expressed as

$$s(t) = \underbrace{a_0 \prod_{i=1}^{P}(1 - p_i e^{j\Omega t}) \prod_{i=1}^{Q}(1 - \frac{1}{q_i^*} e^{j\Omega t})}_{\text{MinP}} \underbrace{\frac{\prod_{i=1}^{Q}(1 - q_i e^{j\Omega t})}{\prod_{i=1}^{Q}(1 - \frac{1}{q_i^*} e^{j\Omega t})}}_{\text{AllP}} \tag{17}$$

Again, this representation is analogous to the unique decomposition of a linear system into its minimum phase and all-pass parts. Hence, similar to $s(t)$'s representation given by Eq. 12, the signal can be expressed as a product of a MinP signal and an AllP signal as

$$s(t) = A_c e^{\alpha(t)+\beta(t)+j\left(\hat{\alpha}(t)+\hat{\beta}(t)\right)} e^{j\left(\omega_c t - 2\hat{\beta}(t)\right)} \quad , \tag{18}$$

where $e^{j\left(\omega_c t - 2\hat{\beta}(t)\right)}$ is an AllP signal. The main point is that an analytic signal can be characterized by its positive envelope (in the traditional sense) and by a positive IF (of its AllP part) rather than by its usual IF (phase-derivative). Intuitively, this characterization tells us to define a signal's IF as the derivative of that part of its phase which is left over after removing the contribution due to the signal's log-envelope (specifically its Hilbert transform) from the original phase. Thus, given $s(t) = a(t)e^{j\phi(t)}$ we define its positive IF (PIF) as

$$\text{PIF of } s(t) = \frac{d\left(\phi(t) - \widehat{\ln a}(t)\right)}{dt} \quad . \tag{19}$$

In [10], Loughlin and Tracer have addressed splitting of a signal's phase into two parts using a different approach. In the following section, we propose a generalized AM-FM demodulator to compute the PIF of an analytic signal. Remarkably, the algorithm does not require explicit computation of either the logarithm or the Hilbert transform. More details will be given in [11].

5. A GENERAL AM-FM DEMODULATOR

The proposed AM-FM demodulator, shown in Figure 1, consists of two parts. In the first part we model the envelope of the signal $s(t)$ (see Eq.(18)) by minimizing the energy of $e(t)$, defined as

$$\int_0^T |e(t)|^2 dt = \int_0^T |s(t)h(t)|^2 dt \tag{20}$$

where $h(t) = 1 + \sum_{k=1}^{H} h_k e^{jk\Omega t}$ is a voltage controlled oscillator (VCO) output. The minimization is achieved by choosing the coefficients, h_ks; $\Omega = 2\pi/T$. One may recognize this signal envelope modeling method as the analog of the linear prediction (autocorrelation) method well known in spectral analysis [12]. We call our method Linear Prediction in Spectral Domain or LPSD. Similar to the MinP

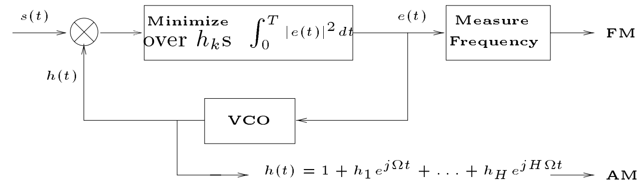


Figure 1: Block diagram for the AM-FM demodulator; AM in Figure corresponds to the traditional envelope of $1/h(t)$ (*i.e.* $1/|h(t)|$) while FM denotes the positive Instantaneous Frequency of $s(t)$'s All-Phase component.

property of the prediction error filter used in linear prediction ([12]), it can be shown that minimizing $\int_0^T |e(t)|^2 dt$ will result in a $h(t)$ that is a MinP signal (having all its zeros inside the unit-circle). The significance of this property is that $h(t)$'s log-envelope and phase are Hilbert transforms. Because the error minimization is performed to approximate $s(t)$'s envelope, if the value of H is chosen sufficiently large, then $h(t)$ will be given by

$$h(t) \approx e^{-(\alpha(t)+\beta(t))} e^{-j\left(\hat{\alpha}(t)+\hat{\beta}(t)\right)} \quad . \tag{21}$$

Thus, $\frac{1}{h(t)}$ is the desired approximation to $s_{\text{MinP}}(t)$. Consequently the error signal $e(t)$ will be

$$e(t) \approx A_c e^{j\left(\omega_c t - 2\hat{\beta}(t)\right)} \quad , \tag{22}$$

and hence is an approximation to the AllP signal; the PIF can be obtained as $\frac{\dot{e}(t)}{|e(t)|}$ or $\frac{d\angle e(t)}{dt}$.

For example, consider a signal having 8 zeros (shown in Fig. 2(a)) and a magnitude-spectrum as shown in Fig. 2(b). The signal is sampled at 16 kHz, has 9 harmonically related complex sinusoids of frequencies 0 Hz, 200 Hz, upto 1.6 kHz, with amplitudes 1, 3.37, 3.42, 9.45, 15.76, 5.4, 3.72, and 1.5 respectively, and whose respective phases (in radians) are 0, −0.3, −1.3, −3.1, 2.8, 2.7, −1.3, −0.9, and −0.6. The signal's duration is 10 msecs (corresponding to two periods of 160 samples). We compute this signal's PIF using LPSD with model order $H = 15$. Observe that while the signal's IF (Fig. 2(c)) is negative at times, the PIF (solid line in Fig. 2(d)) is always positive; dashed-dotted line in Fig. 2(d) corresponds to the true PIF. Although the example considered was of a periodic signal, in general the log-envelope and PIF can be estimated by applying LPSD over successive windowed portions of any given signal.

6. CONCLUSION

We showed that an analytic signal viewed through a window of T seconds can be uniquely decomposed into its MinP and AllP components; it can thus be represented by its (MinP's) positive envelope and by its (AllP's) PIF. The algorithm we proposed for this purpose does not require explicit computation of either the logarithm or the Hilbert transform.

7. REFERENCES

[1] D. Gabor, "Theory of communication," *Proceedings of the IEE*, vol. 93, pp. 429–457, 1946.

[2] L. Cohen, *Time-Frequency Analysis*. Englewood Cliffs, NJ: Prentice Hall, 1995.

[3] J. Dugundji, “Envelopes and preenvelopes of real waveforms,” *IRE Trans. Inform. Theory*, vol. 4, pp. 53–57, 1958.

[4] M. S. Gupta, “Definition of instantaneous frequency and frequency measurability,” *American Journal of Physics*, vol. 43, pp. 1087–1088, Dec. 1975.

[5] B. Boashash, “Estimating and interpreting the instantaneous frequency of a signal,” *Proceedings of the IEEE*, vol. 80, pp. 520–568, Apr. 1992.

[6] H. B. Voelcker, “Towards a unified theory of modulation part I: Phase-Envelope relationships,” *Proceedings of the IEEE*, vol. 54, no. 3, pp. 340–354, 1966.

[7] M. Poletti, “The homomorphic analytic signal,” *IEEE Transactions on Signal Processing*, vol. 45, pp. 1943–1953, 1997.

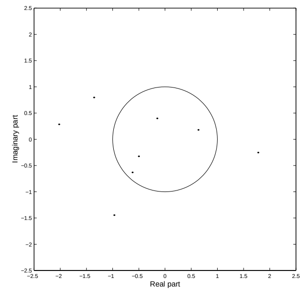
[8] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

[9] A. Rao and R. Kumaresan, “On instantaneous frequency of multicomponent signals,” *IEEE Transactions on Signal Processing*, submitted, July 1997.

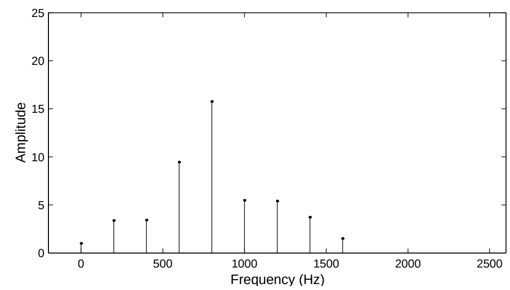
[10] P. J. Loughlin and B. Tacer, “On the amplitude- and frequency-modulation decomposition of signals,” *Journal of the Acoustical Society of America*, vol. 100(3), pp. 1594–1601, Sept. 1996.

[11] R. Kumaresan and A. Rao, “Phase decomposition of signals,” *Journal of the Acoustical Society of America*, Submitted May 1997.

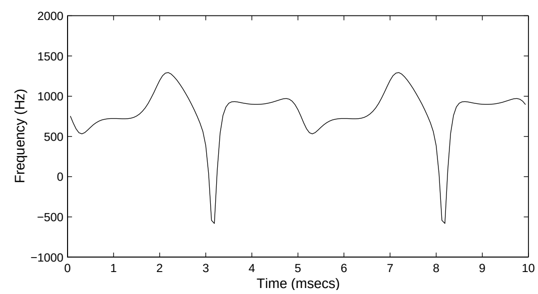
[12] S. M. Kay, *Modern Spectral Estimation : Theory and Application*. Englewood Cliffs, NJ: Prentice Hall, 1987.



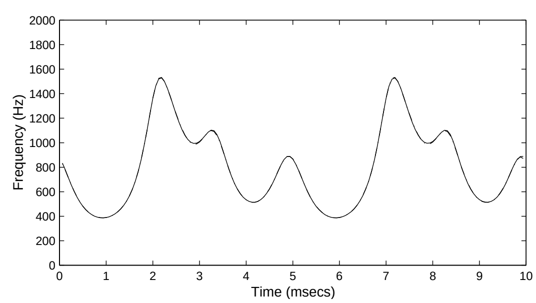
(a)



(b)



(c)



(d)

Figure 2: We consider a MixP signal consisting of 8 zeros (shown in Fig. 2(a)) having a magnitude-spectrum as shown in Fig. 2(b). The signal's IF (Fig. 2(c)) is negative around 3 msecs and 8 msecs. Its PIF displayed in Fig. 2(d) is always positive; the PIF estimated using LPSD (solid line in Fig. 2(d)) and the true PIF (dashed-dotted line in Fig. 2(d)) match closely.