LOCALLY OPTIMUM DETECTORS FOR DETERMINISTIC SIGNALS IN MULTIPLICATIVE NOISE

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ABSTRACT

This paper addresses the problem of detecting deterministic signals in multiplicative noise. The multiplicative noise model is appropriate for modelling coherent imaging systems such as SAR and LASER. Locally Optimum (LO) detectors are derived for any arbitrary multiplicative noise distribution. The gamma and generalized Gaussian distributions are studied in detail. We also introduce an extension of the generalized Gaussian density to include asymmetry. The performance of the LO detectors is studied and compared with that of the linear correlation detector. The paper gives insight into the influence of the tail length of the noise distribution on the detection power.

1. INTRODUCTION

Signal detection and estimation in noise is an important problem in many areas of applied science and electrical engineering. The additive noise model has been used mostly. Relatively much less attention has been given to non-additive noise models. This paper addresses the detection of deterministic signals in multiplicative noise.

Multiplicative noise is often characterized as signal dependent in the sense that it vanishes in the absence of signal. Such a situation is encountered in Doppler-radar [1], sonar [4] and fading communication channels [13]. This detection model has recently been studied in [2] and [11]. For harmonic signals, an adaptive line enhancer-based detection is studied in [6].

By contrast, this paper studies a noise model where the multiplicative noise is not signal-dependent

$$X_n = (1 + \theta s_n)\, Y_n, \qquad n = 1, ..., N \tag{1}$$

$\{s_n\}$ is a known deterministic sequence, $\theta \geq 0$ is the unknown signal strength parameter, and $\{Y_n\}$ is a multiplicative noise sequence with finite mean μ and variance σ^2. It is assumed that $1 + \theta s_n > 0$, for $n = 1, ..., N$, and $\{Y_n\}$ is a sequence of independent and identically distributed random variables with probability density function (pdf) f_y.

For the additive noise model, the observations are given by

$$X_n = \theta s_n + Y_n, \qquad n = 1, ..., N \tag{2}$$

Models (1) and (2) have the common property that the observed signal is noise when $\theta = 0$, i.e. $X_n = Y_n$. The mean of Y_n in (2) can be assumed zero without loss of generality (wlog). The non-zero mean can always be removed from X_n. However, μ cannot be assumed zero wlog. for the multiplicative noise model (1). More specifically, it will be shown that the Detection Power (DP) is an increasing function of the coherent-to-non-coherent signal power ratio μ^2/σ^2.

Model (1) is appropriate for modelling images produced by coherent radiation imagery systems, e.g. Synthetic Aperture Radar (SAR). In this context, $n = (i, j)$ represents the spatial coordinates (range and azimuth) of the image pixels. The multiplicative noise (speckle) reduces the detectability of ground targets and thus decreases the accuracy of feature classification. An important problem in speckle imagery is the detection of intensity changes at object boundaries, i.e. edge detection between two regions with different reflectivities [3]. The resulting SAR image intensity can be modeled well by (1) when the signal $\{s_n\}$ is a unit step function. When no texture profile change occurs in a test window, $\theta = 0$ and the observed SAR image is a stationary process.

Model (1) is also appropriate for a special class of nonstationary signals, the so-called scaled processes (or uniformly modulated time series [10]). Such processes are known to be useful for approximating seismic reflectivity data. Detecting abrupt variance jump in zero mean random signals is a special case for model (1).

The paper is organized as follows. The next section derives Locally Optimum (LO) detectors for model (1). Section 3 analyses their detection performances. Comparisons between the LO detectors and the Linear Correlation (LC) detector are also investigated. Section 4 presents conclusions.

2. LOCALLY OPTIMUM DETECTOR

The optimal Neyman-Pearson (NP) detector for model (1) requires known θ, even for Gaussian noise. Thus, it is not possible to derive Uniformly Most Powerful (UMP) detectors (which are optimal for all values of θ in an anticipated interval). UMP detectors do exist for the additive noise model (2) under the Gaussian assumption [7]. One approach to this problem is to replace θ by its maximum likelihood estimator [5]. The resulting detector is the well-known Generalized Likelihood Ratio (GLR) detector. However, this technique performs poorly for small θ and small sample-size. Moreover, GLR detectors may be too complex to implement, especially for non Gaussian noise. An alternative solution develops LO detectors for which θ can be unknown. In addition to simple implementation, LO detectors are optimum for the challenging case of vanishing SNR $(\theta \to 0)$. Hence, the detection problem considered is

$$\mathcal{H}_0: \ \theta = 0 \qquad versus \qquad \mathcal{H}_1: \ \theta > 0$$

A LO detector maximizes the slope of the likelihood ratio at $\theta = 0$ while keeping a fixed False Alarm (FA) probability. According to the generalized NP lemma and under mild regularity conditions [7], the LO test is

$$\frac{\left.\frac{\partial p(X/\mathcal{H}_1)}{\partial \theta}\right|_{\theta=0}}{p(X/\mathcal{H}_0)} \begin{cases} > \kappa \Rightarrow \mathcal{H}_1 \\ < k \Rightarrow \mathcal{H}_0 \end{cases} \tag{3}$$

$p(X/\mathcal{H}_i)$ is the likelihood function of the observation vector $X = [X_1, ..., X_N]$ under hypothesis $\mathcal{H}_i$, and κ is a threshold chosen to achieve the desired FA probability.

Using the statistical independence of the noise sequence, the joint pdf of the observation vector X in (1) is

$$p(\mathbf{x}/\theta) = \prod_{n=1}^{N} \frac{1}{1+\theta s_n} f_y\left(\frac{x_n}{1+\theta s_n}\right) \tag{4}$$

where $\mathbf{x} = [x_1, ..., x_N]$. The test statistic for LO detector is found, after some calculations, to be

$$T_{LO}(X) = \sum_{n=1}^{N} s_n X_n g_y(X_n). \tag{5}$$

where $g_y(.)$, a memoryless nolinearity, is given by

$$g_y(.) = -\frac{f_y'(.)}{f_y(.)} \tag{6}$$

In (6) f_y' denotes the derivative of f_y. The following studies the gamma and the Generalized Gaussian (GG) distributions.

2.1. Gamma Density

The pdf of the speckle intensity is known to be a negative-exponential distribution [3]. Multiple looks are averaged incoherently to reduce speckle. Thus, the resulting speckle intensity is gamma distributed for L-look SAR, i.e.

$$f_y(y) = \left(\frac{L}{\mu}\right)^L \frac{y^{L-1}}{,(L)} e^{-Ly/\mu} U(y) \tag{7}$$

where $,(L)$ is the gamma function of order L and $U(.)$ is the unit step function ($U(y) = 1$ if $y \geq 0$ and $U(y) = 0$ otherwise).

The UMP does not generally exist since the test statistic for the optimal NP detector depends upon θ:

$$T_O(X) = \sum_{n=1}^{N} \frac{s_n x_n}{1+\theta s_n} \tag{8}$$

The memoryless nonlinearity for density (7) is given by

$$g_y(x) = \frac{L}{\mu} - \frac{L-1}{x} \tag{9}$$

Using (5), the LO detector is found to coincide with the LC detector:

$$T_{LO}(X) = T_{LC}(X) = \sum_{n=1}^{N} s_n x_n \tag{10}$$

Note that the LC detector is not just LO but also UMP for constant signals. Thus, for edge detection in SAR images, the LC detector is UMP under gamma noise modelling.

2.2. Generalized Gaussian Density

The tail length is one of the most important parameters characterizing noise distributions. A good model to describe densities with variable tail lengths is the GG distribution [8, 7]. Let $V_n = Y_n - \mu$ and let $f_v(.)$ be the corresponding univariate pdf, which is modeled by

$$f_v(v) = p_{gg}(v) := \frac{\alpha}{2\beta,(1/\alpha)} exp\left(-\left|\frac{v}{\beta}\right|^{\alpha}\right), \tag{11}$$

where $,(.)$ is the gamma function, α ($\alpha > 0$) and β ($\beta > 0$) are the shape and size parameters repectivelly. For $\alpha = 2$, $f_v(v)$ is the Gaussian density, whereas for $\alpha = 1$, we obtain the Laplace distribution. The GG model has been shown to fit disturbance distributions in many applications of signal and image processing. For example, impulsive noise can be modeled by the GG density with small values of α.

For the Gaussian additive noise model, the LC detector is also UMP. However, even under the Gaussian assumption, it is generally not possible to obtain UMP detectors for the multiplicative noise model (1). The sufficient statistic for the optimal NP detector in the Gaussian multiplicative noise case requires knowledge of θ:

$$T_O(X) = \sum_{n=1}^{N} \frac{s_n}{1+\theta s_n}\left[\frac{2+\theta s_n}{1+\theta s_n} X_n^2 - 2\mu X_n\right] \tag{12}$$

where the subscript $_O$ refers to the optimal NP detector.

The memoryless nonlinearity for the GG density in (11) is:

$$g_v(v) = \frac{\alpha}{\beta^\alpha} |v|^{\alpha-1} \operatorname{sgn}(v), \tag{13}$$

Using (5), the LO detector test statistic is

$$T_{LO}(X) = \sum_{n=1}^{N} s_n X_n |X_n - \mu|^{\alpha-1} \operatorname{sgn}(X_n - \mu) \tag{14}$$

For zero mean multiplicative noise, the LO detector is also UMP for constant signals, i.e. $s_n = c$ for $n = 1, ..., N$ where $c = \pm 1$, w.l.g.. Constant signals are useful for detecting abrupt changes. In this case, the UMP detector test statistic is

$$T_{UMP}(X) = c\sum_{n=1}^{N} |X_n|^{\alpha} \tag{15}$$

The GG density is a good model for symmetric distributions having variable sharpness. To describe asymmetric-densities, we next propose an extension of the GG distribution.

2.3. Extension of the GG pdf to include asymmetry

Asymmetric GG models have been recently proposed in [9] and [12]. In [9], the proposed model allows the left and right tail lengths to be different. In [12], the asymmetry is introduced by different values of the left and right variances whereas the tail length is the same for both sides. Both of theses extensions depend upon three parameters only. A more general model should depend on four parameters, left and right variances and shape parameters. Below, we introduce a new extension of the GG model and show its ability to overcome modelling limitations of the existing extensions.

The new extension of the GG pdf, which will be called asymmetric GG (AGG), is given by

$$p_{AGG}(v) = \begin{cases} \frac{\alpha_l\alpha_r}{\alpha_r\beta_l\Gamma(1/\alpha_l)+\alpha_l\beta_r\Gamma(1/\alpha_r)} exp\left(-\left|\frac{v}{\beta_l}\right|^{\alpha_l}\right), & v < 0 \\ \frac{\alpha_l\alpha_r}{\alpha_r\beta_l\Gamma(1/\alpha_l)+\alpha_l\beta_r\Gamma(1/\alpha_r)} exp\left(-\left|\frac{v}{\beta_r}\right|^{\alpha_r}\right), & v \geq 0 \end{cases} \tag{16}$$

where

$$\beta_l = \sigma_l\left[\frac{\Gamma(1/\alpha_l)}{\Gamma(3/\alpha_l)}\right]^{1/2}; \ \beta_r = \sigma_r\left[\frac{\Gamma(1/\alpha_r)}{\Gamma(3/\alpha_r)}\right]^{1/2} \tag{17}$$

α_l and α_r are the left and right shape parameters, σ_l and σ_r are the left and right variances. Our model reduces to the one proposed in [12] when $\alpha_l = \alpha_r$.

The kth-order moment of the AGG density is found to be

$$m_k = \frac{-\sigma_l^k}{1+A}\frac{,\left(\frac{k+1}{\alpha_l}\right)}{,\left(\frac{1}{\alpha_l}\right)}\left(\frac{,\left(\frac{1}{\alpha_l}\right)}{,\left(\frac{3}{\alpha_l}\right)}\right)^{\frac{k}{2}}$$

$$+\frac{\sigma_r^k}{1+1/A}\frac{\Gamma\left(\frac{k+1}{\alpha_r}\right)}{\Gamma\left(\frac{1}{\alpha_r}\right)}\left(\frac{\Gamma\left(\frac{1}{\alpha_r}\right)}{\Gamma\left(\frac{3}{\alpha_r}\right)}\right)^{\frac{k}{2}} \tag{18}$$

where

$$A=\frac{\alpha_l}{\alpha_r}\frac{\sigma_r}{\sigma_l}\left(\frac{\Gamma\left(\frac{1}{\alpha_l}\right)}{\Gamma\left(\frac{1}{\alpha_r}\right)}\right)^{3/2}\left(\frac{\Gamma\left(\frac{3}{\alpha_r}\right)}{\Gamma\left(\frac{3}{\alpha_l}\right)}\right)^{1/2} \tag{19}$$

The median is equal to zero, while the mean is generally nonzero. Since V_n has to be zero mean, we consider the centered AGG (CAGG) density which is defined as

$$f_v(v)=p_{CAGG}(v):=p_{AGG}(v-m_1) \tag{20}$$

where m_1 is given in (18). Now, the mean is zero and the median is equal to m_1. An interesting class of the CAGG model is that of densities having equal values for the mean and median, i.e. $m_1=0$. This implies that the tail length and variance changes occur at the mean value. The constraint $m_1=0$ implies that the CAGG has only three degrees of freedom; the fourth parameter is a function of the three other parameters. It is worth noting that the models proposed in [9] and [12] fail in modelling asymmetric densities whose mean and median are identical. Indeed, for those models, the constraint $m_1=0$ implies symmetry.

The memoryless nonlinearity of the CAGG in (20) is

$$g_{LO}(v)=\begin{cases}\frac{\alpha_l}{\beta_l^{\alpha_l}}\left|v-m_1\right|^{\alpha_l-1}, & v<m_1\\ \frac{\alpha_r}{\beta_r^{\alpha_r}}\left|v-m_1\right|^{\alpha_r-1}, & v\geq m_1\end{cases} \tag{21}$$

The LO test statistic (5) is then obtained as

$$\begin{aligned}T_{LO}(X)=&\frac{\alpha_l}{\beta_l^{\alpha_l}}\sum_{X_n<m_1+\mu}s_nX_n\left[-(X_n-m_1-\mu)\right]^{\alpha_l-1}\\ &+\frac{\alpha_r}{\beta_r^{\alpha_r}}\sum_{X_n\geq m_1+\mu}s_nX_n\left[X_n-m_1-\mu\right]^{\alpha_r-1}\end{aligned} \tag{22}$$

3. PERFORMANCE ANALYSIS

3.1. Finite-Sample Performance

Analytical derivations of the finite sample-size detection performance are in general not tractable. However, it is possible to get an explicit closed-form for finite sample-size performance in the important case of constant amplitude signals and gamma noise density. In this case, the LO detector, which is the LC detector, is UMP.

The distribution of test statistic (10) is the following gamma density

$$p_T(t/\theta)=\left(\frac{NL/\mu}{1+\theta}\right)^{NL}\frac{t^{NL-1}}{\Gamma(NL)}e^{-\frac{Lt}{\mu(1+\theta)}}U(t) \tag{23}$$

when $s_n=1, n=1,...,N$. The LC detector DP is, for a given strength parameter θ (>0), given by

$$P_D=\int_{\kappa+LN_\Delta}^{\infty}p_T(t/\theta)dt \tag{24}$$

where κ is the threshold relative to the fixed FA probability. The DP is an increasing function of L, as expected. The speckle reduction and the image profile enhancement are increasing functions of the number of SAR looks. Numerical evaluations of the DP versus L provides insight into the number of SAR looks required to obtain some desired DP. For example, to detect an intensity change of amplitude $\theta=0.1$ with power $P_D>0.9$ and FA probability $P_{FA}=0.01$, 30 looks of $N=20$ pixels SAR images need to be averaged. The finite sample-size performance for the (asymmetric) GG density is carried out using Monte-Carlo experiments. The results will be presented elsewhere.

3.2. Asymptotic Performance

The efficacy of a test which decides between $\theta=0$ and $\theta>0$ and which is based on a test statistic T is given by (under some regularity conditions [7])

$$\xi=\lim_{N\to\infty}\frac{\left[(\partial/\partial\theta)\,E\left\{T(X)/\mathcal{H}_1\right\}|_{\theta=0}\right]^2}{N\,\mathrm{var}\left\{T(X)/\mathcal{H}_0\right\}} \tag{25}$$

where E and var denote the statistical mean and variance respectively. Using Taylor expansion of $T_{LO}(X)$ at $\theta=0$ and under the following mild assumptions

$$yf_y(y)\underset{y\to\infty}{\to}0,\quad y^2f_y'(y)\underset{y\to\infty}{\to}0, \tag{26}$$

the efficacy for the LO detector is found to be

$$\xi_{LO}=P_s\eta_y \tag{27}$$

where

$$\eta_y=E\left\{\left(Y_n\frac{f_y'(Y_n)}{f_y(Y_n)}\right)^2\right\}-1 \tag{28}$$

and P_s is the signal power

$$P_s=\lim_{N\to\infty}\frac{\left(\sum_{n=1}^N s_n^2\right)}{N} \tag{29}$$

The proof of this result is skiped here and will be given elsewhere. The performance of the LO detector is a monotonic increasing function of η_y. In order to obtain an explicit expression for ξ_{LO} as a function of μ, we rewrite η_y as

$$\eta_y=\mu^2I_0+2\mu I_1+I_2, \tag{30}$$

where I_0, I_1 and I_2 are the following "Fisher informations for location"

$$I_0=E\left\{\left(\frac{f_v'(V_n)}{f_v(V_n)}\right)^2\right\};\qquad I_1=E\left\{V_n\left(\frac{f_v'(V_n)}{f_v(V_n)}\right)^2\right\};$$

$$I_2=E\left\{V_n^2\left(\frac{f_v'(V_n)}{f_v(V_n)}\right)^2\right\}-1 \tag{31}$$

I_1 vanishes for symmetric noise distributions. It is worth noting that for the additive noise model (2), the efficacy of the LO detector is P_sI_0.

The LO detector is also asymptoticaly optimal. The asymptotic optimality can be proved using the heuristic approach developed in [7] for the additive noise model. Hence, ξ_{LO} is the asymptotic optimal performance for our detection problem.

For the CAGG distribution I_0, I_1 and I_2 are found to be

$$\begin{aligned}I_0&=\frac{\alpha_l^2}{2\sigma_l^2}\frac{\Gamma\left(2-1/\alpha_l\right),\left(3/\alpha_l\right)}{\Gamma^2(1/\alpha_l)}+\frac{\alpha_r^2}{2\sigma_r^2}\frac{\Gamma\left(2-1/\alpha_r\right),\left(3/\alpha_r\right)}{\Gamma^2(1/\alpha_r)}\\ I_1&=-\frac{\alpha_l^2}{2\sigma_l}\frac{\left(\Gamma\left(3/\alpha_l\right)\right)^{1/2}}{\Gamma^{3/2}(1/\alpha_l)}+\frac{\alpha_r^2}{2\sigma_r}\frac{\left(\Gamma\left(3/\alpha_r\right)\right)^{1/2}}{\Gamma^{3/2}(1/\alpha_r)}+m_1I_0\\ I_2&=\frac{\alpha_l+\alpha_r}{2}+2m_1I_1-m_1^2I_0\end{aligned} \tag{32}$$

For the gamma distribution, the efficacy equals P_sL, where L is the number of SAR looks.

3.3. Influence of the Tail Length

Below, we limit our study to the symmetric GG density, for which η_y is obtained as

$$\eta_y = \frac{\mu^2}{\sigma^2} \frac{\alpha^2 ,\, (2-1/\alpha) ,\, (3/\alpha)}{,\,^2(1/\alpha)} + \alpha, \tag{33}$$

The parameter η_y and the DP are increasing functions of μ^2/σ^2. To get insight into the influence of α on the DP, numerical evaluations of η_y versus α are carried out for different values of μ^2/σ^2. The results are depicted in figure 1. It is worth reminding that the detection performance for the additive noise model is worst for Gaussian noise ($\alpha = 2$). This result is no longer true for model (1). The value of α which minimizes η_y depends upon μ^2/σ^2. The detection performance is similar to that of the additive noise model for large values of μ^2/σ^2 (i.e. worst for Gaussian noise). That is, the multiplicative noise density minimizing the DP tends to the Gaussian distribution when $\mu^2 >> \sigma^2$.

3.4. Comparison with the LC detector

The Asymptotic Relative Efficiency (ARE) is often employed as a measure of relative asymptotic detection performance [7]. The efficacy of the LC detector is given by

$$\xi_{LC} = P_s \frac{\mu^2}{\sigma^2} \tag{34}$$

Thus, the LC detector fails for detecting signals in zero mean multiplicative noise. The ARE of the LC detector with respect to the LO detector is then

$$ARE_{LC,LO} := \frac{\xi_{LC}}{\xi_{LO}} = \frac{\mu^2}{\sigma^2 \eta_y} \tag{35}$$

$ARE_{LC,LO}$ is an increasing function of μ^2/σ^2. Numerical evaluations of $ARE_{LC,LO}$ show that the LO detector significantly outperforms the LC detector for small μ^2/σ^2 and/or large deviation of α from 2. The following limit is also obtained for the GG distribution

$$ARE_{LC,LO} \underset{\frac{\mu^2}{\sigma^2} \to \infty}{\longrightarrow} \frac{,\,^2(1/\alpha)}{\alpha^2 ,\, (2-1/\alpha) ,\, (3/\alpha)} \tag{36}$$

Note that the limit in (36) is unity for Gaussian noise and equal to 0.5 for $\alpha = 1$.

4. CONCLUSIONS

This paper has addressed signal detection in multiplicative noise. It is generally not possible to obtain UMP detectors, even under Gaussian noise distribution. LO detectors are then developed for arbitrary multiplicative noise densities. The gamma and (asymmetric) GG densities are studied in detail. The paper gives an initial insight into LO detection schemes for SAR imagery system. The LC detector is shown to be UMP under gamma noise models for edge detection in SAR images. Furthermore, it is shown that the LO detectors outperform the LC detector for symmetric multiplicative noise densities. In some cases, the performance improvement is very large. A detailed study of the detection performance for the asymmetric GG distribution for both additive and multiplicative noise models will be reported elsewhere.

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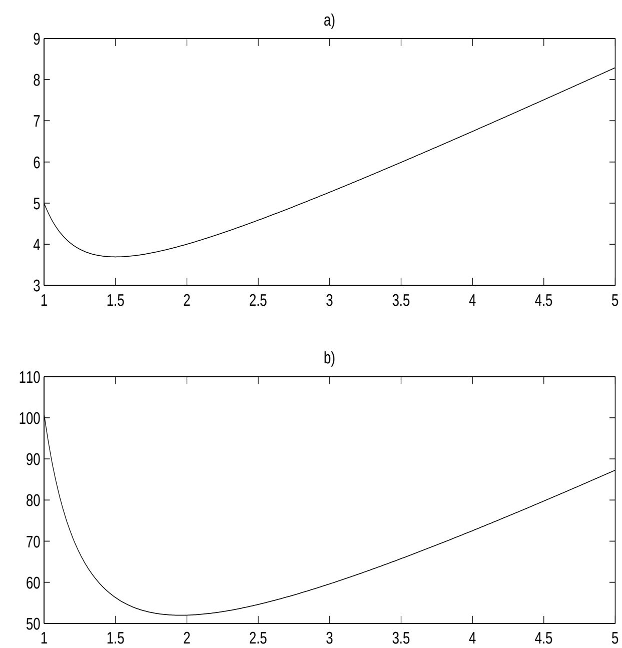

Figure 1. Plots of η_y versus α. *a)* $\mu^2/\sigma^2 = 2$, *b)*$\mu^2/\sigma^2 = 50$.