DISCRETE-COEFFICIENT LINEAR-PHASE PROTOTYPES FOR PR COSINE-MODULATED FILTER BANKS

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ABSTRACT

In this paper, a method for the design of perfect reconstruction (PR) linear-phase prototypes for cosine-modulated filter banks with discrete coefficients is presented. Such prototypes are of great interest for efficient hardware implementations. The design procedure is based on a subspace approach that allows to linearly combine PR prototype filters in such a way that the resulting filter also is a PR prototype. Within a given subspace the weights of the optimal linear combination can be easily computed via an eigenanalysis. The filter design is carried out iteratively, while the PR property is guaranteed throughout the design process. No non-linear optimization routine is needed.

1. INTRODUCTION

Cosine-modulated filter banks are very popular in signal processing, because of their efficiency [1][2][3][4]. In this class of filter banks, all analysis and synthesis filters are modulated versions of a single prototype. The implementation of the complete filter bank depicted in Figure 1 then only requires the implementation of polyphase components of the prototype and of the modulation, which itself can be efficiently realized via FFTs.

The quality of the filter bank for a given application mainly depends on the properties of the prototype. For the design of the prototype, which will be denoted as $P(z)$, we can follow various strategies. A method that structurally guarantees the perfect-reconstruction (PR) property of the filter bank is the use of lattice factorizations [5]. For this method a good starting point is required, because we have to optimize angles in a cascade of lattices and the relations between the angles and the impulse response are highly nonlinear. A second method that typically is less sensitive to the starting point is the quadratic-constraint algorithm [6]. This method does not inherently guarantee PR, but the PR requirements can be satisfied with arbitrary accuracy. A simple but efficient iterative design method for practically useful near PR prototypes was presented in [7]. This method is based on the older pseudo-QMF ideas ([1]) rather than on the PR constraints ([2][3][4]), so that on principle only near PR prototypes can be designed.

In this paper a new design method is proposed that, like the lattice factorization, guarantees the PR property. The optimization is performed iteratively by optimizing linear combinations of impulse responses within suitable linear subspaces. Throughout the filter design process, no non-linear optimization routine is required. However, non-linear optimization may be used in order to achieve further improvements.

Filters with integer-valued coefficients are very much desirable, because they allow the efficient implementation of filter banks. The simplest way to design such an integer prototype is to quantize the coefficients of a given prototype. Clearly, when doing this, the PR property of the prototype gets lost and we will need relatively many bits in order to achieve at least an almost perfect reconstruction. In this paper a different way is proposed that keeps the PR property throughout the design process while dealing only with integers. The design process can be initialized with a simple rectangular window. From such a starting point, PR integer prototypes with desired coefficient wordlengths can be designed in a simple way.

2. COSINE-MODULATED FILTER BANKS

In cosine-modulated filter banks the impulse responses of the analysis and synthesis filters, $h_k(n)$ and $g_k(n)$, can be derived from a single prototype $p(n)$ in the following way [2][3][4]:

$$\begin{aligned} h_k(n) &= p(n)\,\sqrt{\frac{2}{M}}\,\cos\left[\frac{\pi}{M}\,\left(k+\frac{1}{2}\right)\,\left(n-\frac{L-1}{2}\right)+\phi_k\right], \\ g_k(n) &= h_k(L-1-n), \end{aligned} \tag{1}$$

where $n=0,\dots,L-1,\;\; k=0,\dots,M-1$, and $\phi_k=\frac{\pi}{4}(-1)^k$. M is the number of channels, which is assumed to be even. The filter length, L, is chosen as $L=2mM,\ m>1$, where m is an integer. Furthermore, critical subsampling is considered.

If the low-pass prototype $p(n)$ satisfies the symmetry condition

$$p(n)=p(L-n-1), \quad n=0,\dots,\frac{L}{2}-1, \tag{2}$$

and if the polyphase components

$$P_k(z)=\sum_{n=0}^{m-1} z^{-n}\,p_k(n), \qquad p_k(n)=p(2nM+k), \tag{3}$$

of the prototype satisfy the condition

$$\tilde{P}_k(z)\,P_k(z)\,+\,\tilde{P}_{M+k}(z)\,P_{M+k}(z)=1, \quad k=0,\dots,\frac{M}{2}-1, \tag{4}$$

then the filter bank is paraunitary, that is, the filter bank has the PR property and, moreover, it provides a unitary transform [2][3][4][5]. Terms with a tilde accent in (4), like $\tilde{P}_k(z)$, are the z-transforms of sequences $p_k^*(-n)$, which are derived from $p_k(n)$ by complex conjugation and time reversion. We restrict us here to the case of real-valued prototypes (i.e. $p_k^*(n) = p_k(n)$). A product of the type $\tilde{P}_k(z) P_k(z)$ in (4) is nothing but the z-transform of the autocorrelation sequence of $p_k(n)$.

3. OPTIMALITY CRITERION

When a prototype has a high stopband attenuation (low stopband energy) it will give good performance in a wide range of applications. Therefore we use the stopband energy as the performance measure and formulate the optimization problem using the following Rayleigh quotient:

$$C(\boldsymbol{p}) = \frac{\boldsymbol{p}^T\ \boldsymbol{V}_s\ \boldsymbol{p}}{\boldsymbol{p}^T\ \boldsymbol{p}} \stackrel{!}{=} \min \tag{5}$$

The vector $\boldsymbol{p}$ contains the unknown filter coefficients, $\boldsymbol{p} = [p(0), p(1), \ldots, p(L-1)]^T$ and $\boldsymbol{V}_s$ is a weighting matrix defined by

$$\boldsymbol{p}^T\ \boldsymbol{V}_s\ \boldsymbol{p} = \int_{\text{stopband}} |P(\omega)|^2\ d\omega, \tag{6}$$

where $P(\omega)$ is the Fourier transform of $p(n)$.

Unfortunately, the optimal solution to (5) will not satisfy condition (4), that is, the optimal filter in the sense of (5) cannot be used as a prototype for a PR cosine-modulated filter bank, and condition (4) must be included in the optimization process. In the sequel a new subspace-based approach for this task will be presented.

4. SUBSPACE APPROACH

Let us consider the optimality criterion (5), and let us assume that we have a set of basis vectors for the design of our optimal prototype $p(n)$. That is, let us assume that we can write $\boldsymbol{p}$ in the following form, where matrix $\boldsymbol{F}$ contains the basis and $\boldsymbol{\alpha}$ contains the coefficients to be optimized:

$$\boldsymbol{p} = \boldsymbol{F}\,\boldsymbol{\alpha}. \tag{7}$$

If all linear combinations of the columns of matrix $\boldsymbol{F}$ lead to a PR prototype $\boldsymbol{p}$, we can formulate the optimization problem as

$$C(\boldsymbol{\alpha}) = \frac{\boldsymbol{\alpha}^T\ \boldsymbol{U}_s\ \boldsymbol{\alpha}}{\boldsymbol{\alpha}^T\ \boldsymbol{U}_p\ \boldsymbol{\alpha}} \stackrel{!}{=} \min \tag{8}$$

where $\boldsymbol{U}_s = \boldsymbol{F}^T \boldsymbol{V}_s\, \boldsymbol{F}$, $\boldsymbol{U}_p = \boldsymbol{F}^T \boldsymbol{F}$, and we can solve (8) for the optimal $\boldsymbol{\alpha}$ in an unrestricted way. Thus, the solution is given by the eigenvector $\boldsymbol{\alpha}$ corresponding to the minimum eigenvalue λ of the generalized eigenvalue problem

$$\boldsymbol{U}_s\ \boldsymbol{\alpha} = \lambda\, \boldsymbol{U}_p\ \boldsymbol{\alpha}. \tag{9}$$

As will be shown below, the drawback of the subspace method considered above is that it cannot be complete. In other words, we cannot find a linear subspace of the space $\mathbb{R}^L$ where all prototypes $p(n)$ of length L satisfying (4) lie in the subspace, and where all linear combinations of the basis vectors of this subspace are PR prototypes [8]. However, in the following we will see that the principle of a PR basis approach works within a two-dimensional subspace.

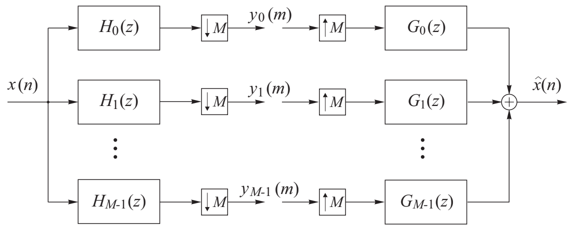

Figure 1. Critically subsampled M-channel filter bank.

4.1. Bases for two-dimensional subspaces

For any given linear-phase prototype $a(n)$ we can construct a second linear-phase PR prototype $b(n)$ in such a way that any linear combination of $a(n)$ and $b(n)$ results in a prototype for a PR cosine-modulated filter bank (up to some scaling factor γ). This means

$$\tilde{P}_k(z)\ P_k(z)\ +\ \tilde{P}_{M+k}(z)\ P_{M+k}(z) = \gamma, \qquad \gamma \neq 0, \tag{10}$$

where

$$P_k(z) = \alpha_1\, A_k(z) + \alpha_2\, B_k(z) \tag{11}$$

Herein, $A_k(z)$ and $B_k(z)$ satisfy the PR condition (4):

$$\begin{aligned} \tilde{A}_k(z)\ A_k(z)\ +\ \tilde{A}_{M+k}(z)\ A_{M+k}(z) = 1, \\ \tilde{B}_k(z)\ B_k(z)\ +\ \tilde{B}_{M+k}(z)\ B_{M+k}(z) = 1. \end{aligned} \tag{12}$$

Given the filters $A_k(z)$ and $B_k(z)$, we can seek for the best linear combination of these filters in the sense of (8), where

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} a(0) & a(1) & \ldots & a(L-1) \\ b(0) & b(1) & \ldots & b(L-1) \end{bmatrix}^T. \tag{13}$$

Clearly, we cannot find the global optimum this way, but we can use this procedure iteratively. We will return to this point in Section 5.

Let $A_k(z),\ k = 0, \ldots, 2M-1$, be the polyphase components of a given prototype filter. In order to construct filters $B_k(z),\ k = 0, \ldots, 2M-1$, in such a way that (10) is satisfied, we combine (10) and (11). Since $A_k(z)$ and $B_k(z)$ satisfy (12), this leads to the following condition:

$$\begin{aligned} \tilde{A}_k(z)\, B_k(z) + A_k(z)\, \tilde{B}_k(z) + \tilde{A}_{M+k}(z)\, B_{M+k}(z) \\ + A_{M+k}(z)\, \tilde{B}_{M+k}(z) = c = \text{const.}, \end{aligned} \tag{14}$$

where $c = (\gamma - \alpha_1^2 - \alpha_2^2)/(\alpha_1 \alpha_2)$. Given $A_k(z)$ and $A_{M+k}(z)$, (14) is nothing but an underdetermined linear set of equations for $B_k(z)$ and $B_{M+k}(z)$. This means, we can choose any solution to (14) for $c \neq 0$ and add any further solution from the nullspace ($c = 0$). Since $B_k(z) = A_k(z)$ is a simple (but valid) solution to (14), it becomes clear that we should look for solutions in the nullspace only, that is, we should solve (14) for $c = 0$:

$$\begin{aligned}&\tilde{A}_k(z)\, B_k(z) + A_k(z)\, \tilde{B}_k(z) + \tilde{A}_{M+k}(z)\, B_{M+k}(z) \\ &+A_{M+k}(z)\, \tilde{B}_{M+k}(z) = 0, \qquad k = 0, \ldots, \tfrac{M}{2} - 1.\end{aligned} \tag{15}$$

The following solutions to (15) can be identified:

$$B_k(z) = \pm A_{M+k}(z), \quad B_{M+k}(z) = \mp A_k(z), \tag{16}$$

and

$$B_k(z) = \pm z^{-(m-1)}\, \tilde{A}_{M+k}(z), \quad B_{M+k}(z) = \mp z^{-(m-1)}\, \tilde{A}_k(z). \tag{17}$$

Note that (15) can be interpreted as an orthogonality relation between the vectors $[\tilde{A}_k(z), A_k(z), \tilde{A}_{M+k}(z), A_{M+k}(z)]$ and $[\tilde{B}_k(z), B_k(z), \tilde{B}_{M+k}(z), B_{M+k}(z)]$.

4.2. Linear Independence

In the construction of our polyphase filters $B_k(z)$ we have the choice to take the solution from (16) or from (17), and we can also choose the signs. This means that (16) and (17) define a whole class of impulse responses $b(n)$. The number of impulse responses $b(n)$ that can be constructed via (16) and (17) by trying all permutations is 2^M. One half of this set can be generated from the other half by a simple sign change, so that we can expect no more than 2^{M-1} filters $b(n)$ with different frequency responses. Such a set of 2^{M-1} filters will be denoted as $\mathcal{B}$. All elements of $\mathcal{B}$ satisfy the PR condition (12), and they are orthogonal to $a(n)$ in the sense of (15). However, evaluating (15) for different solutions of (16) and (17) (instead of $A_k(z)$ and $B_k(z)$) shows that we have no orthogonality within $\mathcal{B}$. This means that the subspace approach only allows the construction of twodimensional subspaces with the property that any linear combination of the elements of the subspace yields a PR prototype.

4.3. Number of linearly independent filters in $\mathcal{B}$

It was already mentioned above that a set $\mathcal{B}$ consists of 2^{M-1} filters. However, it can be shown that only M of these solutions are linearly independent (the proof is given in [8]). This fact can be used in order to reduce the computational cost of the filter optimization.

In the special case of $L = 2M$, the equations (16) and (17) are equivalent, because our polyphase filters $A_k(z)$ are just scalars ($A_k(z) = \tilde{A}_k(z) = a(k)$), and we only have $2^{\frac{M}{2}-1}$ elements of $\mathcal{B}$ with different magnitude frequency responses and $\frac{M}{2}$ linearly independent vectors in $\mathcal{B}$.

5. FILTER OPTIMIZATION AND RESULTS

5.1. Filter Design

The filter-design method consists of the following steps:

1. Given a PR prototype $a(n)$, we construct the set $\mathcal{B}$ of filters $b(n)$ from (16) and (17). Alternatively, we construct a subset of $\mathcal{B}$, denoted as $\mathcal{B}'$, that contains M linearly independent elements.
2. For all filters from $\mathcal{B}$ (or $\mathcal{B}'$) we solve the optimization problem (8), and we select the best candidate. Note that the eigenvalue problem (9), which gives the solution to (8), only contains matrices of size 2×2, so that simple analytical solutions for the eigenvalues and eigenvectors can be provided.
3. The optimal linear combination of $a(n)$ and the selected $b(n)$ is taken as a new initial solution for Step 1. The process is continued until convergence is achieved.

Note that one important feature of the process is the fact that the PR property is preserved throughout the optimization.

Basically, the design-method described above leads to prototypes with infinite-precision coefficients. However, one remarkable feature of the orthogonality relations (16) and (17) is the fact that the prototype $B(z)$ essentially has the same coefficients as the prototype $A(z)$: The filter $B(z)$ is composed from flipped and/or sign-changed polyphase components of the filter $A(z)$. This means, if we start the design with a PR filter $A(z)$ having only integer coefficients and if we also use integer weights α_1 and α_2 in (11) then also the linear combination of $A(z)$ and $B(z)$ will have integer coefficients.

The simplest choice for the initial filter with integer coefficients is a rectangular window which just provides the polyphase transform of the input signal. That is, in an M-band setting only M subsequent coefficients are equal to one and all other (formally introduced) coefficients are zero. In order to design PR filters with integer coefficients it then remains to quantize the weights:

$$\alpha_k := \text{round}(\nu\, \alpha_k), \qquad k = 1, 2, \qquad \nu \in \mathbb{R}.$$

For a full search over $\mathcal{B}$ we have to take 2^{M-1} filters into account in each iteration step. For increasing M this means an exponentially increasing computation effort. An alternative is to search only over the subset $\mathcal{B}'$ of linearly independent filters.

On the one hand, the fact that we only have M linearly independent elements in $\mathcal{B}$ is an advantage, because we can reduce the computation effort of the optimization process by performing the search over $\mathcal{B}'$. On the other hand, for constant M and increasing filter length we have a constant number of linearly independent filters in $\mathcal{B}$. This means, for long filters we have some mismatch between the number of filter coefficients that have to be optimized and the degree of freedom. However, for moderate filter lengths the convergence properties turned out to be excellent.

5.2. Design Examples

Table 1 shows prototype coefficients with different wordlengths for $M = 8$ bands and filter lengths $L = 4M$. Because of symmetry, only the first $2M$ coefficients are listed. The frequency responses of the filters from Table 1 are shown in Figure 2. The comparison to the ELT prototype from ([9]) shows that (in relation to the wordlengths) the prototypes have very good performances. For example, the filter (c) with coefficients in the range $[-1, 8]$ has an acceptable frequency response for image coding purposes while the implementation cost can be kept extremely low.

Prototype coefficients for $M = 4$, 8, and 16 bands can be downloaded from [10].

6. CONCLUSION

In this paper, a novel method for the design of prototypes for PR cosine-modulated filter banks has been presented. The approach is iterative, while the PR property is preserved throughout the optimization. Each iteration step consists of the computation of optimal linear combinations of impulse responses. The linear combinations have to be performed for filters from suitable linear subspaces. The computational cost of the filter optimization is extremely low. The most important feature of the new design method is the fact that it allows the design of PR prototypes with integer-valued coefficients which are desirable for efficient hardware realizations.

7. REFERENCES

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Table 1
Perfect reconstruction prototypes for 8-band filter banks with integer coefficients

	$p(n)$					
n	(a)	(b)	(c)	(d)	(e)	(f)
0			-1	-6	-72	-2190
1			-1	-4	-97	-1901
2			0	0	-41	-1681
3			0	-6	-48	-426
4			0	7	56	497
5			0	0	62	2542
6			2	8	194	3802
7			2	17	204	6205
8		1	4	24	390	9678
9		1	4	33	524	13197
10		1	6	41	656	16359
11		1	6	48	774	19398
12	1	2	7	56	903	22631
13	1	2	7	62	992	24738
14	1	2	8	66	1048	26394
15	1	2	8	68	1105	27421
$C(\boldsymbol{p})$	211.7	40.4	11.9	5.8	2.8	1.6

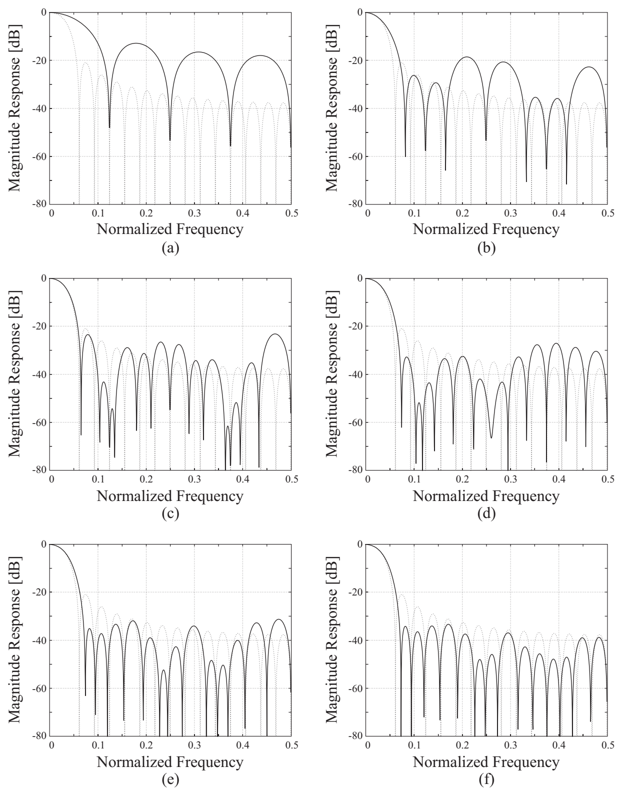

Figure 2. Frequency responses for the 8-channel prototypes from Table 1. For comparison purposes the frequency response of the ELT prototype is depicted with dotted lines.