OPTIMAL LOADING FACTOR FOR MINIMAL SAMPLE SUPPORT SPACE–TIME ADAPTIVE RADAR

Y. L. Kim

Dept. of Electrical Engg.
Polytechnic University
Six Metrotech Center
Brooklyn, NY 11201

S. U. Pillai

Dept. of Electrical Engg.
Polytechnic University
Six Metrotech Center
Brooklyn, NY 11201

J. R. Guerci

SAIC
Air Defense Technology
4001 N. Fairfax Drive, Suite 400
Arlington, VA 22203

ABSTRACT

A major issue in space-time adaptive processing (STAP) for airborne moving target indicator (MTI) radar is the so-called sample support problem. Often, the available sample support for estimating the interference covariance matrix leads to severe rank deficiency, thereby precluding STAP beamforming based on the direct sample matrix inversion (SMI) method. The intrinsic interference subspace removal (ISR) technique, which is a computationally and analytically useful form of diagonally loaded SMI method, is derived here. It covers from Hung-Turner Projection (HTP) algorithm to matched filter according to the loading factor. Also the optimum loading factor which gives the maximum signal-to-interference-plus-noise ratio (SINR) is derived here from the viewpoint of singular value decomposition of the covariance matrix. The simulation results with synthetic data show that the maximum SINR indeed coincides with the proposed optimum loading factor in various data sample situations.

1. INTRODUCTION

A major issue in space-time adaptive processing (STAP) for airborne moving target indicator (MTI) radar is the so-called sample support problem. The returned space-time snapshot signal may consist of a target echo and interferences such as jammer, clutter and thermal noise and can be represented as

$$\mathbf{x}_i = \alpha_t \mathbf{a}_t + \mathbf{c}_i \,, \tag{1}$$

where α_t and $\mathbf{a}_t$ are the complex attenuation factor and target steering vector respectively associated with the spatial and doppler parameters θ and ω_d of the moving target, and $\mathbf{c}_i$ represents the total interference signal. The output of a STAP beamformer, which is the complex weighted sum of each snapshot element, is usually compared to a threshold to determine the presence or absence of the moving target based on the Neyman-Pearson criterion. In the point-doppler estimation problem, the optimum weight vector which maximizes the output signal-to-interference-plus-noise ratio (SINR)

$$\rho = \frac{P_t |\mathbf{w}^* \mathbf{a}_t|^2}{\mathbf{w}^* R \mathbf{w}} \tag{2}$$

is given by [6, 5]

$$\mathbf{w}_o = R^{-1} \mathbf{a}_t \tag{3}$$

where $R = E\{\mathbf{c}\mathbf{c}^*\}$ represents the interference covariance matrix.

In actual practice, the interference covariance matrix R must be estimated by making use of returns from neighboring range bins to the point of interest. When k independent and identically distributed (i.i.d.) samples are available, the maximum-likelihood estimate (MLE) of R is given by

$$\hat{R}_k = \frac{1}{k} \sum_{i=1}^{k} \mathbf{x}_i \mathbf{x}_i^* = \frac{1}{k} Y_k Y_k^* \,, \tag{4}$$

where $\mathbf{x}_i$ represents the snapshot data for i-th neighboring range bin, and Y_k the total data matrix defined by

$$Y_k \triangleq [\mathbf{x}_1 \,, \mathbf{x}_2 \,, \cdots , \mathbf{x}_k] \,. \tag{5}$$

In that case, the direct sample matrix inversion (SMI) weight vector can be represented as [5]

$$\hat{\mathbf{w}}_o = \hat{R}_k^{-1} \mathbf{a}_t \,, \tag{6}$$

and the number of i.i.d. samples k required for the above inversion must be greater than the dimension of STAP (length of $\mathbf{x}_i$).

2. FAST DIAGONALLY LOADED SMI ALGORITHM

With M and N representing the number of pulses in a coherent processing interval (CPI) and antenna elements respectively, in general the number of samples k

This research was partially supported by the Office of Naval Research under contract N-00014-89-J-1512P-5.

should be greater than or at least equal to MN for $\hat{R}_k$ in (4) to be nonsingular (invertible). However, such a large number of samples will introduce nonstationary information, where the statistics are different from that around the target range bin and hence the MLE may not converge. With a small number of samples, however, stationarity becomes more meaningful, but $\hat{R}_k$ does become singular. Many approaches are possible to accommodate such a singular situation. In strong interference cases, one of the simplest method is to add a scaled version of the identity matrix producing diagonal loading [1]

$$\hat{R}_\delta = \hat{R}_k + \delta I_{MN} = \frac{1}{k} Y_k Y_k^* + \delta I_{MN} , \tag{7}$$

where the small positive real number δ is used mainly for the invertability of $\hat{R}_\delta$. This imposes various interesting questions: Is there an optimum value of δ for diagonal loading? Is it possible to eliminate diagonal loading without essentially compromising the performance? Surprisingly, as we show below, the answer to these questions are both positive, and it offers a reasonable solution when the number of samples are sufficiently small compared to the size of the covariance matrix.

To obtain the inverse of this positive definite matrix $\hat{R}_\delta$ in (7), the matrix inversion identity

$$[P^{-1} + MQ^{-1}M^*]^{-1} = P - PM[M^*PM + Q]^{-1}M^*P \tag{8}$$

can be used by letting

$$P \triangleq \frac{1}{\delta} I_{MN} , \quad Q \triangleq kI_k \quad \text{and} \quad M \triangleq Y_k . \tag{9}$$

This gives the loaded covariance matrix inverse to be [3]

$$\hat{R}_\delta^{-1} = \frac{1}{\delta}[I_{MN} - Y_k(Y_k^* Y_k + k\delta I_k)^{-1} Y_k^*] . \tag{10}$$

The scale factor $\frac{1}{\delta}$ can be ignored since it doesn't effect the intrinsic structure of the weight vector and hence we obtain [3, 4]

$$\mathbf{w}_\delta = \delta \hat{R}_\delta^{-1} \mathbf{a}_t = \mathbf{\Phi}_k(\delta)\mathbf{a}_t , \tag{11}$$

where

$$\mathbf{\Phi}_k(\delta) \triangleq \delta \hat{R}_\delta^{-1} = I_{MN} - Y_k(Y_k^* Y_k + k\delta I_k)^{-1} Y_k^* . \tag{12}$$

Note that (11)–(12) has the same computational complexity as the Hung-Turner Projection (HTP) method that is known to be the simplest and fastest projection algorithm [2]. Moreover, since

$$\lim_{\delta \to 0} \mathbf{w}_\delta = [I_{MN} - Y_k(Y_k^* Y_k)^{-1} Y_k^*]\mathbf{a}_t \tag{13}$$

and

$$\lim_{\delta \to \infty} \mathbf{w}_\delta = \mathbf{a}_t , \tag{14}$$

$\mathbf{w}_\delta$ agrees that the Hung-Turner projection [2] for zero diagonal loading. Similarly as $\delta \to \infty$, $\mathbf{w}_\delta$ also approaches the quiescent vector $\mathbf{a}_t$ that represents ordinary beamforming. This raises an interesting question: Is there an optimum δ with respect to maximizing the output SINR?

3. OPTIMUM LOADING FACTOR

The optimum positive real value for the diagonal loading factor can be obtained in terms of the eigenvalues of the estimated covariance matrix $\hat{R}_k$. Towards this, let [1]

$$\hat{R}_k = \sum_{i=1}^{MN} \lambda_i \mathbf{v}_i \mathbf{v}_i^* \tag{15}$$

represent the eigen decomposition of $\hat{R}_k$. It can be shown that $\mathbf{w}_\delta$ in (11) can be expressed as

$$\mathbf{w}_\delta = \mathbf{a}_t - \sum_{i=1}^{MN-1} c_i (\mathbf{v}_i^* \mathbf{a}_t)\mathbf{v}_i . \tag{16}$$

where

$$c_i \triangleq \frac{\lambda_i - \lambda_{MN}}{\lambda_i + \delta} . \tag{17}$$

Because of the rank deficiency of the clutter covariance matrix, it is reasonable to assume (for high clutter-to-noise ratio) that $\lambda_r \gg \lambda_{r+1}$ with r representing the clutter subspace dimensionality [6]. Thus the clutter subspace and the noise only subspace are assumed to be spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$ and $\{\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \cdots \mathbf{v}_{MN}\}$ respectively. In the absence of imperfections (i.e., in the exact nonsingular covariance case), (16) reduces to

$$\mathbf{w}_o = \mathbf{a}_t - \sum_{i=1}^{r} \left(1 - \frac{\sigma^2}{\lambda_i}\right) (\mathbf{v}_i^* \mathbf{a}_t)\mathbf{v}_i , \tag{18}$$

since in that case $\lambda_i = \sigma^2$, $i \geq r+1$. Obviously, the optimally loaded scenario should try to simulate the situation in (18), and hence ideally $c_{r+1} = c_{r+2} = \cdots = c_{MN}$ should equal zero. Since λ_r can be considered as the smallest one among the eigenvalues in the clutter subspace and λ_{r+1} is the largest one among those in the noise subspace, in a finite sample situation the proper loading factor should maximize c_r and minimize c_{r+1} [4].

For the positive definite loaded matrix, the loading factor should be larger than $-\lambda_{MN}$, and accordingly,

[1] The upper limit in the summation in (15) will be MN if $k > MN$. Otherwise, it will be k. In that case, $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_{MN} = 0$.

the range of coefficient c_i is $[0, 1)$ and $c_i = 0$ only when $\lambda_i = \lambda_{MN}$. Hence, the proper loading factor will make c_r approach unity and c_{r+1} to zero. Define the error function by the summation of c_{r+1} and $1 - c_r$, i.e.,

$$f_e(\delta) \stackrel{\triangle}{=} (1 - c_r) + c_{r+1} \, , \tag{19}$$

then we have

$$\left. \frac{\partial f_e(\delta)}{\partial \delta} \right|_{\delta = \delta_o} = 0 \tag{20}$$

at

$$\delta_o = -\lambda_{MN} + \sqrt{(\lambda_r - \lambda_{MN})(\lambda_{r+1} - \lambda_{MN})} \, , \tag{21}$$

satisfying the minimum. Notice that δ_o always satisfies $\lambda_r < \delta_o < \lambda_{r+1}$ for small values of λ_{MN} .

Figs.1–3 show the output SINR vs. the loading factor normalized by the noise variance. In the simulations, the number of pulses per CPI was chosen to be 16 ($M = 16$), the number of array elements to be 14 ($N = 14$). This gives the full degree of freedom covariance matrix size to be 224×224. Fig.1 corresponds to the full rank case where $k > MN$, and Figs.2–3 are for singular situations with $k > r$ and $k < r$ respectively.

Fig.1(a) shows the performance of loaded SMI in (7) normalized by the direct SMI since the sample covariance matrix is invertible in that case. (note that ISR is not applicable here). Fig.1(b) shows the distribution of eigenvalues $\lambda_1 \to \lambda_{224}$, and they span from 85 dB to -5 dB. Notice that $\lambda_{33} \to \lambda_{224}$ cluster together around 0 dB, indicating the noise subspace cutoff point, and hence, we may choose the clutter subspace dimension r in this case to be 32. The loading factor in (21) with $r = 32$ corresponds to the region where the performance is indeed maximum (dashed line in Figs.1(a)–(b)). Notice that peak point in Fig.1(a) falls between λ_{32} and λ_{33} in Fig.1(b). Fig.2 corresponds to the situation where the number of samples is less than MN and greater than the clutter subspace dimension r. In this case $\lambda_{32} \to \lambda_{80}$ cluster together in Fig.2(b), and hence r is chosen to be 31 from the eigenvalue spread in Fig.2(b). Since $\lambda_{MN} = 0$ in this case, the loading factor is represented by (dashed line in Fig.2(b))

$$\delta_o = \sqrt{\lambda_r \lambda_{r+1}} \, . \tag{22}$$

Once again from Figs.2(a)–(b), performance is optimum in this range (between λ_{31} and λ_{32}).

Fig.3 corresponds to a severely undersampled case with $k = 10$, the number of samples being less than the actual rank of the clutter subspace. In this case, since $\lambda_{MN} = \lambda_r = \lambda_{r+1} = 0$, we have $\delta_o = 0$ that corresponding to the Hung-Turner projection method, and peak performance is indeed attained for that value.

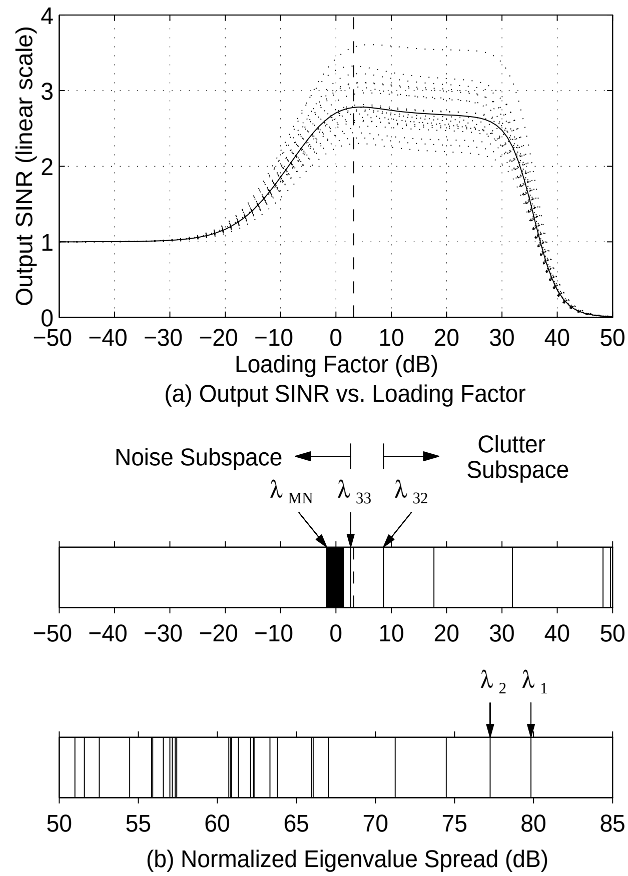

Figure 1: Full rank situation with $k = 300 (> MN)$. (a) Output SINR normalized by SMI vs. loading factor normalized by the noise power. The dotted lines show 20 independent simulations and the solid line their average. The vertical dashed line is the estimated optimum loading factor when $r = 32$. (b) Normalized eigenvalue spread of the sample covariance matrix.

4. CONCLUSION

If the available number of data samples is less than the size of covariance matrix, the diagonal loaded situation covers from Hung-Turner projection to the quiescent weight vector according to the loading factor, and the optimum loading factor which maximizes the output SINR is derived and applied to the clutter situation. When the number of samples is less than the clutter subspace dimension r, the optimum loading factor is shown to be zero and corresponds to the Hung-Turner projection method. Otherwise when $r < k < MN$, the optimum loading factor lies between the smallest clutter subspace eigenvalue and the largest noise subspace eigenvalue and it maximizes the output SINR.

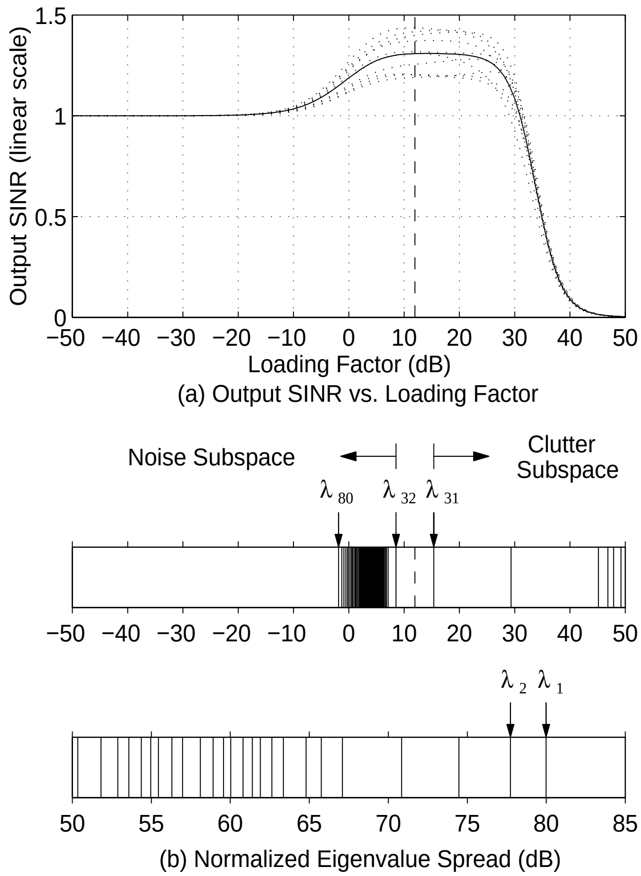

Figure 2: Singular case with $k = 80\,(r < k < MN)$. (a) Output SINR normalized by HTP vs. loading factors normalized by the noise power. The dotted lines show 20 independent simulations and the solid line their average. The vertical dashed line is the estimated optimum loading factor when $r = 31$. (b) Normalized eigenvalue spread of the sample covariance matrix.

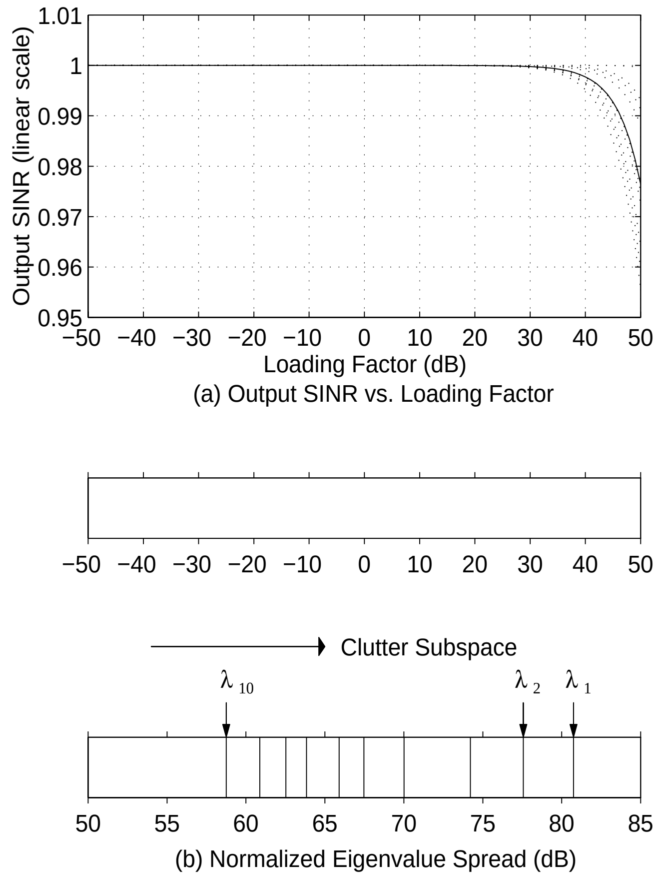

Figure 3: Singular case with $k = 10\,(< r)$. (a) Output SINR normalized by HTP vs. loading factors normalized by the noise power. The dotted lines show 20 independent simulations and the solid line their average. Since $\lambda_{MN} = \lambda_r = \lambda_{r+1} = 0$, we have $\delta_o = 0$ and ISR gives the best performance. (b) Normalized eigenvalue spread of the sample covariance matrix.

5. REFERENCES

[1] Carlson, B.D., "Covariance Matrix Estimation Errors and Diagonal Loading in Adaptive Arrays", *IEEE Transactions on Aerospace and Electronic Systems*, vol. 24, no. 4, July 1988.

[2] Hung, E.K.L., and Turner, R.M., "A Fast Beamforming Algorithm for Large Arrays", *IEEE Transactions on Aerospace and Electronic Systems*, vol. 19, No. 4, July 1983.

[3] Pillai, S.U., and Kim, Y.L., and Guerci, J.R., "A New Implicit Subspace Interference Removal Technique for Space-Time Adaptive Radar", *Proceedings of the Fifth Annual Workshop on Adaptive Sensor Array Processing*, MIT Lincoln Laboratories, Lexington, MA, March 12-14, 1997.

[4] Pillai, S.U., and Kim, Y.L., and Guerci, J.R., "New Techniques for Minimal Sample Support Space-Time Adaptive Radar", Under preparation.

[5] Reed, I.S., Mallet, J.D., and Brennan, L.E., "Rapid Convergence Rate in Adaptive Antennas", *IEEE Transactions on Aerospace and Electronic Systems*, vol. 10, no. 6, November 1974.

[6] Ward, J., "Space-Time Adaptive Processing", Technical Report 1015, MIT Lincoln Laboratory, Lexington, MA, December 1994.