AN IRREGULAR SAMPLING ALGORITHM ADAPTED TO THE LOCAL FREQUENCY CONTENT OF SIGNALS AND THE CORRESPONDING ON-LINE RECONSTRUCTION ALGORITHM

Voicu Filimon

Daimler Benz Forschungszentrum, Wilhelm-Runge-Str. 11, D-89081 Ulm, Germany

ABSTRACT

Description of signals using wavelet transforms leads to useful time-frequency localization and possible signal compression. Based on the Discrete Wavelet Transform (DWT) an adaptive sampling algorithm in the discrete time domain is constructed, by finding an univocal relation between the signal's samples and the non-zero transform coefficients of its DWT. Reconstruction is performed through repeated projections of an approximation of the initial signal based on the arriving samples, into the original signal's subspace, using the Neumann method of inverting bounded operators. Both adaptive sampling and reconstruction are on-line because of the finite support of the analyzing wavelets.

1. INTRODUCTION

Compression of discrete signals using the Fourier Transform (FT) followed by decimation according to the so determined spectral support is already common practice. However this method bears the disadvantages of inexactness due to the inherent truncation of the signal when calculating the FT. It is also difficult to implement on-line since good results are subject to relatively long portions of signal taken into consideration. The good localization in frequency provided by the FT is paid for with poor localization in time. Using other systems of orthogonal functions providing a better compromise between time and frequency localization, could lead to a locally adapted decimation algorithm. The decision of whether a sample should or should not be transmitted to make perfect reconstruction possible, could be taken "on the spot" using a few adjacent values of the signal, and at the same time according to the local frequency content, thus enabling compression by way of adaptive sampling.

Such a system of orthogonal base functions is provided by compactly supported wavelets, generated for example with the two-channel iterated system in Fig.1 using FIR filters. Due to their way of generation [1] [3] [4], the wavelets decompose the frequency domain in constant quality-factor sub-bands. The Fourier coefficients of the DWT performed by the analysis (left) side of the structure in Fig.1, are an indication of the instantaneous frequency localization of the signal. If the two filters have impulse responses h and g, then the orthogonal base functions analyzing the signal, for example in an N iteration structure are ($\uparrow 2$ stands for scaling by 2 using interpolation) :

$$\{(g),(h*g\uparrow 2),...,(h*h\uparrow 2*...*g\uparrow 2^{N-1}),(h*...*h\uparrow 2^{N-1})\} \qquad (1)$$

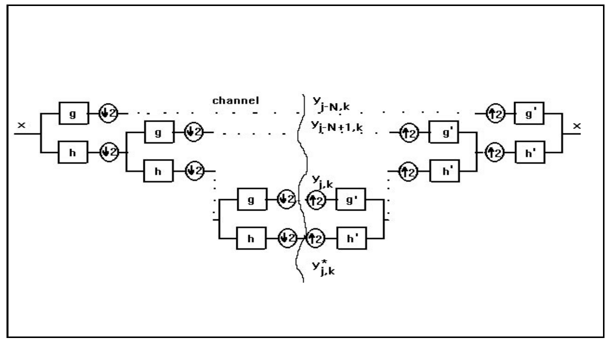

Figure 1. Iterated two-channel filterbank for multi-resolution analysis

With augmenting number of iterations, the discrete sequences in rel. (1) approximate increasingly better their continuous domain wavelet counterparts [1]. The filters we further use, are generated by the parametrization in [3] of the poliphase matrix H_P , where U_0 is ½ [1 1; 1 -1], and **v** = [a b]^T with $a^2+b^2=1$:

$$H_P=\begin{bmatrix} h_0+z^{-1}\cdot h_2 & h_1+z^{-1}\cdot h_3 \\ g_0+z^{-1}\cdot g_2 & g_1+z^{-1}\cdot g_3 \end{bmatrix}=U_0\left\{\left[I-V\cdot V^T\right]+z^{-1}\cdot V\cdot V^T\right\} \qquad (2)$$

We can describe the filters as:

$$\begin{bmatrix} h \\ g \end{bmatrix}=\begin{bmatrix} A & B \end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix} b(b-a) & a(a-b) & a(a+b) & b(a+b) \\ b(a+b) & -a(a+b) & a(a-b) & b(a-b) \end{bmatrix} \qquad (3)$$

with matrices A (first two columns) and B (third and fourth column) defined for calculating purposes. Let us remark that $A^TA+B^TB=1$ and $A^TB=B^TA=0$.

2. ADAPTIVE SAMPLING ALGORITHM

Denote the discrete signal to be adaptively sampled by x and its image in the domain of its DWT coefficients by y and y*. The direct and inverse transformation can be written as:

$$\begin{bmatrix} y^*_{1,2k} \\ y_{1,2k} \end{bmatrix}=A\cdot\begin{bmatrix} x_{2k} \\ x_{2k-1} \end{bmatrix}+B\cdot\begin{bmatrix} x_{2k-2} \\ x_{2k-3} \end{bmatrix} \qquad (4)$$

and:

$$\begin{bmatrix} x_{2k} \\ x_{2k-1} \end{bmatrix}=A^T\cdot\begin{bmatrix} y^*_{1,2k} \\ y_{1,2k} \end{bmatrix}+B^T\cdot\begin{bmatrix} y^*_{1,2k+2} \\ y_{1,2k+2} \end{bmatrix} \qquad (5)$$

Fig 2 illustrates the signal values x_k ,with k the current time index, and the corresponding image values y and y*, where y^*_{ij} are the coefficients corresponding to the analysis with low-pass filter h, and y_{ij} are the coefficients corresponding to the high-pass g filter, i referring to the level of iteration and j to time.

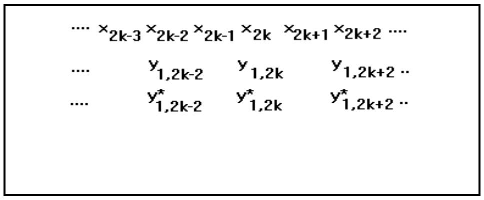

Figure 2. Samples of the source signal x and corresponding DWT coefficients y and y*

The analysis can be iterated on the low-pass branch, transforming the **y*** signal to become, i.e. after 1 iteration:

$$\begin{bmatrix} y^*_{2,4k} \\ y_{2,4k} \end{bmatrix} = A \cdot \begin{bmatrix} y^*_{1,4k} \\ y^*_{1,4k-2} \end{bmatrix} + B \cdot \begin{bmatrix} y^*_{1,4k-4} \\ y^*_{1,4k-6} \end{bmatrix} \qquad (6)$$

with $\mathbf{y_2}$***** and $\mathbf{y_2}$ the image of $\mathbf{y_1}$***.**

Zero coefficients in the DWT transform point to absent frequency components at that particular moment, due to the time-frequency localization properties of the wavelet. Setting $y_{1,2k}$ and $y^*_{1,2k}$ to 0 in eq. (4), and multiplying successively by A^T and B^T gives:

$$\begin{aligned} b \cdot x_{2k} - a \cdot x_{2k-1} &= 0 \\ a \cdot x_{2k-2} + b \cdot x_{2k-3} &= 0 \end{aligned} \qquad (7)$$

This means that if the two **y** coefficients are zero, one can ignore one of the samples $\{x_{2k}$ or $x_{2k-1}\}$ and $\{x_{2k-2}$ or $x_{2k-3}\}$ in transmission, thus obtaining a locally adapted sampling. The remaining samples must uniquely determine the initial signal so the equation system obtained by setting several DWT coefficients to zero must be a determined one. With only one level of iteration, represented by the decomposition in Fig 2, choosing the remaining samples so as to fulfill the condition of non-zero determinant is easy. This is not at all straightforward if the analysis is performed with a multiple iteration structure like in Fig1. After the DWT, the coefficients of type y* (here the low-pass components) are known only for the last level, and signal values x_k and image values y_k and y^*_k depend on one another in a way that the resulting equation system is not of finite order, having a pseudo-diagonal semi-infinite system (transformation) matrix.

We obtain an univocal relation that points to a sample to be left out of the signal **x,** for every coefficient y_{ij} in the image **y** that is of zero value, without affecting the uniqueness of the **x** signal. This is achieved in a few steps:

Theorem 1. Consider the following sequences of real numbers : (x_n), (y_{2n}), (y^*_{2n}), n in Z, satisfying the relations:

$$\begin{aligned} b \cdot x_{2n-1} + a \cdot x_{2n} &= c \cdot y^*_{2n+2} + d \cdot y_{2n+2} \\ -a \cdot x_{2n+1} + b \cdot x_{2n+2} &= -d \cdot y^*_{2n+2} + c \cdot y_{2n+2} \end{aligned} \qquad (8)$$

where $a^2+b^2=1$, a,b !=0, c=(a+b)/sqrt(2), d=(a-b)/sqrt(2) (Obs.: these are eq. (4) and (5) explicited after applying A^T and B^T to them, with the first "1" index in $y_{1,2n+2}$ left aside for simplicity).

If there are k,m in Z with k<m so that x_{2k-1}, x_{2m}, $y_{2k+2}, \dots, y_{2m}$, $y^*_{2k+2}, \dots y^*_{2m}$ are known, than the values $x_{2k}, x_{2k+1}, \dots, x_{2m-1}$ of the signal are uniquely determined .

Corollary 1.: Consider the sequences x, y and y* from Theorem 1 satisfying conditions (8) and the numbers k,m in Z, k<m. If x_{2k-1}, x_{2m} are known, and $y_{2k+2}, \dots, y_{2m}$, $y^*_{2k+2}, \dots y^*_{2m}$ are all zero, then $x_{2k}, x_{2k+1}, \dots, x_{2m-1}$ are uniquely determined.

This is equivalent to saying that $x_{2k}, x_{2k+1}, \dots, x_{2m-1}$ can be neglected at transmission or storing, thus obtaining data compression. But we need an algorithm to uniquely put into correspondence one sample of the signal and one coefficient of the wavelet transform, even when there are no compact sequences of zeros among the coefficients. So further we have:

Theorem 2: Consider the sequences of real numbers from Theorem 1, fulfilling condition (8), and the numbers k,m in Z, k<m. If we know x_{2k-1} and two elements from each of the following sets: $\{x_{2k}, x_{2k+1}, y_{2k+2}, y^*_{2k+2}\}, \dots \{x_{2m-2}, x_{2m-1}, y_{2m}, y^*_{2m}\}$, and x_{2m}, then the elements $x_{2k}, x_{2k+1}, \dots, x_{2m-1}$ are uniquely determined.

Proofs of the theorems are presented in [5]

As a consequence, following sampling procedure can be adopted:

Corollary 2: Under the conditions of Theorem 1, for k<m, and knowing the samples x_{2k-1} and x_{2m}, the following adaptive sampling algorithm leads to a set of samples that uniquely determine the initial signal:

if $y_{2k+2} = 0$ ignore the sample x_{2k+1}
if $y^*_{2k+2} = 0$ ignore the sample x_{2k}

Iterating this corollary to the coefficient sequences y* characterizing the low-pass components of the analyzed signal leads to the following:

Adaptive Sampling Algorithm : Given the conditions of Theorem 1, a signal **x** is uniquely determined, if by analyzing its discrete wavelet transform **y**, each time when:

$$\begin{aligned}
& y_{1,2k} = 0 \quad \textit{one ignores sample:} \quad x_{2k-1} \\
& y_{2,4k} = 0 \quad \textit{one ignores sample:} \quad x_{4k-4} \\
& y_{3,8k} = 0 \quad \textit{one ignores sample:} \quad x_{8k-10} \\
& \dots\dots\dots\dots \\
& y_{N,2^N k} = 0 \quad \textit{one ignores sample:} \quad x_{2^N k-2^N-2^{N-1}+2} \\
& y^*_{N,2^N k} = 0 \quad \textit{one ignores sample:} \quad x_{2^N k-2^{N+1}+2}
\end{aligned}$$

This association of samples of **x** with coefficients in the DWT **y**, which of course is not the only one - but sure not trivial - leading to an uniquely determined signal , is presented in Fig. 3, with running index k set to 0, for easier representation.

Samples belonging to the low-pass sides on lower levels are put into brackets: [y*], because they are not actually present in the DWT, since further DW transformed, but are represented in Fig. 3 to contribute to the better visualization of the algorithm.

In practice the algorithm will ignore all those samples that correspond to coefficients with absolute values smaller than a threshold d>0. The energy of the error signal obtained through reconstruction will be:

$$E = Sy^2_i < d^2 n_{cof} \qquad (9)$$

where the index of summation i runs through the set of coefficients with absolute values smaller than d. This set has the cardinality n_{cof} (the set of sequences of type y and y^* is countable, so it can be renumbered). If we denote by N the length of an interval of the signal **x** and with n_{sa} the number of samples left in this interval after using the procedure described above., then obviously $N=n_{sa}+n_{cof}$, and from eq. (9) we can compute a limit to the error power:

$$P < d^2(1-D) \qquad (10)$$

where $D=n_{sa}/N$ denotes the sampling density. If D is determined successively for parts of the signal **x**, then eq. (10) computes the local mean power of the error. If the reconstruction procedure assures perfect reconstruction, except for the error introduced by those coefficients who where considered to be zero if smaller than d, then rel. (10) represents the noise in the computation of the reconstruction signal-to-noise ratio associated with the algorithm.

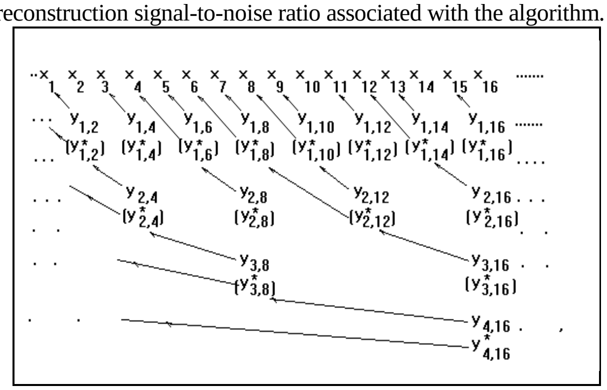

Figure 3. Adaptive sampling algorithm after a 4 iteration-level analysis

Observe also that the algorithm is on-line because of the finite support of the analyzing wavelets, filters h and g in Fig. 1 being FIR.

The algorithm can be used either by transmitting the remaining samples of **x** at the times they occur, or by coding the distance between subsequent samples (also to be done when storing) and transmitting it as well. It should be mentioned that the distance in this discrete setting can be coded without quantization error and on less bits then probably the quality of signal would require, depending on the number of iteration levels.

In either case the positions of the samples and their values are known at reception, and on this information we further build an on-line reconstruction algorithm. Observe that the positions of the arriving samples supply a direct information on which coefficients of the DWT are 0 (respectively were omitted in the process of compression), thus determining the subspace the initial signal **x** was in.

3. ON-LINE RECONSTRUCTION OF AN ADAPTIVELY SAMPLED SIGNAL USING ITERATED PROJECTIONS

The reconstruction of a signal (vector) **x** from its image **y** can be done by solving the system:

$$T\mathbf{x}=\mathbf{y} \qquad (11)$$

where T denotes the linear mapping (matrix) from l^2 to l^2 which corresponds to the DWT . It is a pseudo-diagonal, infinite matrix and the transformation is orthogonal. But the system in eq. (11) is not straightforward to solve (even by known iterative methods): we know some of the components of **x** (the incoming samples) and a number of zero valued coefficients in the image **y**, found out of the distances between the samples, providing a number of equations equal to the number of the components of **x** to be determined. This number is arbitrary and depends on the variable sampling density, that is, it depends on the local frequency behavior of the signal. Also we do not know at reception, which will be the values of **x** to be determined (the unknowns), and in addition the process of reconstruction should be local, addressing a limited number of samples at a time.

To perform the reconstruction we use the Neumann expression of the inverse of a bounded operator, which says that if C is a linear operator with $\|C\|<1$ then [2]:

$$(id - C)^{-1} = \sum_{n=0}^{\infty} C^n \qquad (12)$$

This can be used to obtain for an operator K satisfying:

$$\|x - Kx\| < \gamma\|x\|, \quad \text{with} \quad \gamma < 1 \qquad (13)$$

an iterative procedure of recovering **x** from its image K**x**:

$$x_{n+1} = x_n + K(x - x_n) \qquad (14)$$

If operator K is chosen such as to satisfy rel. (13), and such that it depends only on the incoming samples of the signal **x** and on the distances between them, then K**x** is known at reception and **x** can be recovered.

We obtain such an operator by constructing an approximation of the **x** signal at reception, based only on its samples, and then projecting this approximation into the subspace in which the **x** signal initially was. Following operator is shown in [5] to satisfy rel. (13):

$$K = T^{-1} Q_2 T Q_1 \qquad (15)$$

with:

T: the operator corresponding to the DWT

Q_2: the operator setting to 0 all DWT coefficients which were 0 in the initial signal

Q_1: the operator setting to 0 all values of the signal which were not transmitted since determined to be redundant by the adaptive sampling performed

Hence K applied to **x** turns out to do the following: $Q_1\mathbf{x}$ is the approximation of x consisting only of the samples that were transmitted; $TQ_1\mathbf{x}$ is the projection of the approximation on the whole image space; $Q_2TQ_1\mathbf{x}$ cuts out components that belong to subspaces that where empty in the initial signal; $T^{-1}Q_2TQ_1\mathbf{x}$ returns from the image space to the signal space.

Using K we can reconstruct signal **x** out of its irregularly arriving samples through following iterative algorithm:

$$\mathbf{x}_0 = Q_1\mathbf{x} \qquad (16)$$
$$\mathbf{x}_{n+1} = \mathbf{x}_n + K(\mathbf{x} - \mathbf{x}_n) = \mathbf{x}_n + T^{-1} Q_2T(Q_1\mathbf{x} - Q_1\mathbf{x}_n)$$

It may look as if in order to perform the iterations above we would have to know the entire signal x. But the DWT operator T is performed by the left side of the iterated filter-bank in Fig.1 and its inverse T^{-1} by the right synthesis side, both of which contain FIR filters (the actual computations being done with rel. 4 and 5). This is a local computation and operator K is applied only to parts of the signal, since at a given moment only a few wavelets contribute to its construction. The process in rel. (16) is implemented on-line.

4. EXPERIMENTAL RESULTS AND CONCLUSIONS

Because of the constant-Q frequency decomposition produced by the structure in Fig.1, the adaptive sampling algorithm follows indeed the instantaneous Nyquist rate required by the local frequency content of the signal. That is for periods characterized by high frequency there are many non-zero y_{1k} coefficients on the first or lower level (Fig. 3) , so many samples of **x** have to be transmitted, whereas in the periods of low frequency content only **y** coefficients on the higher levels are non-zero, so fewer values of **x** will be kept. Though utilisable for any kind of signals, the algorithms above are very effective in the case of non-stationary signals, like speech, where the multiresolution analysis promptly detects the shift between frequency bands.

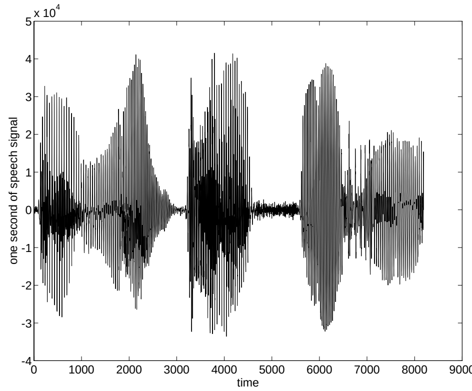

Figure 4. 1 second of speech signal, 8192 values

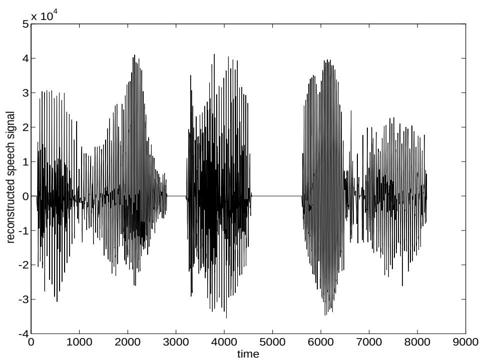

Figure 5. Reconstructed signal, out of 1289 remaining values after adaptive sampling using rel. (17) iterated 9 times

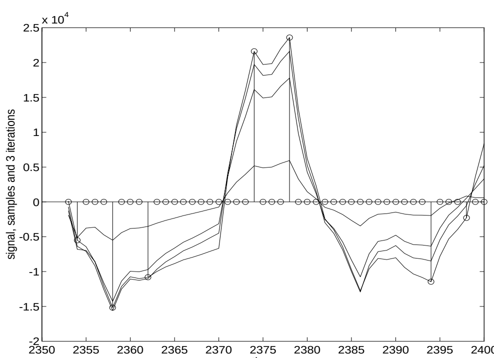

Figure 6. Detail of reconstruction of the signal in Fig.4: signal, remaining samples in this interval, iterations in the on-line reconstruction, illustrating convergence.

Fig. 4 presents one second (8192 values) of a speech signal, sampled adaptively according to the algorithm in Par. 2, using an 8-level wavelet analysis, to result in 1289 remaining samples. The signal was reconstructed using 9 iterations of rel.(17), the result being presented in Fig.5. A detail of reconstruction is presented in Fig.6, where the initial signal, its few remaining samples in that particular time interval after adaptive sampling, and its reconstructed versions after 3, 6 and 9 iterations of rel. (17) can be seen. Correctness of reconstruction and convergence of iterative reconstruction algorithm are illustrated. Fig. 7 contains the local sampling density, D in rel (10), where D was obtained averaging the number of remaining samples over 100 consecutive values of the signal in Fig.4 In this last figure the adaptive character of the algorithm comes fully to expression, and the possibility of estimating an instantaneous frequency suggests itself.

The compression obtained, of about 1 to 8 in Fig. 4 is a usual figure for an 8 level analysis and is a merit of the wavelets and it could be argued that there is no need for transmitting (or storing) the samples of the signal when already having the coefficients of the DW Transform. But this is a method of adaptive sampling and there are cases when having the samples allows a quick estimation of what the signal is about, without performing the reconstruction. The algorithms also provide an easy way of finding and handling pauses in signals.

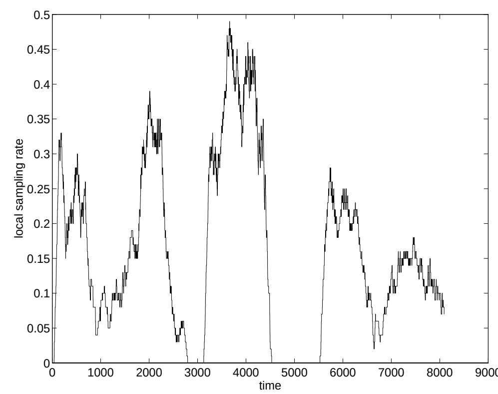

Figure 7. Instantaneous sampling rate for the signal in Fig. 4, obtained by averaging the number of remaining samples over 100 consecutive values of the signal, an indication of the instantaneous frequency.

5. REFERENCES

[1] Daubechies I. "Orthonormal Bases of Compactly Supported Wavelets", Comm. on Pure and Applied Mathematics, Vol.41(1988), 908-996

[2] Feichtinger H.G., Gröchenig K. "Theory and Practice of Irregular Sampling", Preprint

[3] Gopinath R.A., Burrus C.S. "Wavelet Transforms and Filter Banks", in C.K.Chui "Wavelets: A Tutorial in Theory and Application", Academic Press, 1992

[4] Vetterli M. "Wavelets and Filter Banks for Discrete-Time Signal Processing", in M.B.Ruskai "Wavelets and their Applications", Jones a. Bartlett Publishers, 1992

[5] Filimon V. Doctoral Thesis, Technical University of Timisoara, 1994