UNBIASED IDENTIFICATION OF AUTOREGRESSIVE SIGNALS OBSERVED IN COLORED NOISE

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ABSTRACT

Autoregressive (AR) modeling has played an important role in many signal processing applications. This paper is concerned with identification of AR model parameters using observations corrupted with colored noise. A novel formulation of an auxiliary least-squares estimator is introduced so that the autocovariance functions of the colored observation noise can be estimated in a straightforward manner. With this, the colored-noise-induced estimation bias can be removed to yield the unbiased estimate of the AR parameters. The performance of the proposed unbiased estimation algorithm is illustrated by simulation results. The presented work greatly extends the author's previous method that was developed for identification of AR signals observed in white noise.

1. INTRODUCTION

The problem of fitting an autoregressive (AR) model to a data sequence of noisy measurements is of great significance in many application areas of signal processing. It is known that the AR estimator usually shows a high sensitivity to the addition of observation noise to the AR model [2]. This sensitivity, which manifests itself as a severe estimation bias, limits the practical application of AR models in noisy environments. Unfortunately, the literature is rather insufficient in methods for unbiased parameter estimation of noisy AR signals. Among the existing methods, there are the the modified Yule-Walker (MYW) equations method [2], the maximum likelihood (ML) method [6], the recursive prediction error (RPE) method [3], and the modified least-squares (MLS) method [4]. But these methods suffer from various deficiencies.

Recently, the improved least-squares (ILS) method was proposed for identification of AR signals from measurements contaminated by white noise [7], and it seems to be a more promising unbiased estimation algorithm. For example, the ILS method involves less computational costs than the ML, the RPE and the MLS methods. While the MYW method may suffer from numerical instability in on-line implementation, the ILS method is well suited for adaptive estimation. Moreover, the ILS method is not only superior to the MYW and the ML methods in that it can provide a direct estimate of the driving noise variance and the observation noise variance, but also superior to the RPE and the MLS methods in that it has a much simpler scheme for estimating these noise variances. It is shown in [8] that the ILS method can be modified to perform unbiased AR parameter estimation with neither prefiltering noisy data nor making any parameter transformation.

In this paper, we consider the problem of estimating the parameters of AR models in the presence of colored noise. The motivation for this work is that in many practical circumstances, the observation noise, which contaminates the AR signal, may be colored rather than white. So the AR models subject to colored noise can have more widespread signal processing applications. However, this problem has received little attention in the literature. Although the statistical properties of the least-squares (LS) AR estimator in the case of colored noise are analyzed in [5], no unbiased estimation algorithm was proposed there for identifying the AR signal in such noisy situations. We also note that most of the methods mentioned above are limited to the case where there exists only white observation noise.

The aim of this paper is to extend the ILS type method to the general and practical cases of AR modeling with colored-noise-corrupted observations. It is shown that the asymptotic bias in the standard LS estimator of the AR parameters is removable provided that the autocovariance functions of the colored observation noise can be estimated accurately. To this end, an auxiliary parameter vector is introduced into the underlying noisy AR system. A novel formulation of the LS estimator of the introduced parameter vector then provides a direct way for estimating the noise autocovariance functions of interest. The unbiased estimate of the AR parameters follows immediately from application of the bias

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correction principle [1]. Theoretical results are confirmed through computer simulations.

2. PROBLEM FORMULATION

The AR model under study is represented by

$$A(q^{-1})x(t) = v(t) \tag{1}$$

where $\{v(t)\}$ is a stationary zero-mean driving white noise sequence with variance σ_v^2, $\{x(t)\}$ is a true AR signal sequence, q^{-1} is the unit delay operator, and $A(q^{-1}) = 1 - a_1 q^{-1} - \ldots - a_p q^{-p}$. The AR signal $x(t)$ is observed in colored noise as

$$y(t) = x(t) + w(t) \tag{2}$$

where $\{w(t)\}$ is a stationary zero-mean colored measurement noise sequence.

We make the two assumptions on the noisy AR system (1)–(2) under consideration.

A1. $A(q^{-1})$ has all zeros strictly inside the unit circle, and the order of the AR model p is known.

A2. $\{w(t)\}$ is finitely auto-correlated, namely,

$$r_w(k) \equiv E[w(t)w(t-k)] = 0, \text{ for } |k| \geq M \tag{3}$$

where $E[.]$ denotes the expectation operator, and M is a given positive integer. Moreover, $\{w(t)\}$ is statistically independent of $\{v(t)\}$.

Note that Assumption A1 is a standard assumption that guarantees that the AR model is stable. Assumption A2 actually implies that $\{w(t)\}$ is a moving-average (MA) noise sequence, and this assumption conforms to a wide range of practical situations where the observation noise is colored.

Our objective is to arrive at an unbiased estimate of the AR parameters $\{a_i, \ 1 \leq i \leq p\}$ using the colored-noise-corrupted measurements $\{y(t), \ 1 \leq t \leq N\}$, where N denotes the number of data points.

3. IDENTIFICATION ALGORITHM

3.1. Least-Squares Estimator

We first introduce the following notations:

$$\mathbf{a}^\top = [a_1 \ldots a_p] \tag{4}$$

$$\mathbf{y}_t^\top = [y(t-1) \ldots y(t-p)] \tag{5}$$

$$\mathbf{w}_t^\top = [w(t-1) \ldots w(t-p)] \tag{6}$$

The noisy AR system (1)–(2) can be expressed in the linear regression form

$$y(t) = \mathbf{y}_t^\top \mathbf{a} + v(t) + w(t) - \mathbf{w}_t^\top \mathbf{a} \tag{7}$$

The LS estimate of the AR parameter vector **a** minimizes

$$J = E[(y(t) - \mathbf{y}_t^\top \mathbf{a})^2] \tag{8}$$

and is given by

$$\mathbf{a}_{LS} = \mathbf{R}_y^{-1} \mathbf{r}_y \tag{9}$$

where $\mathbf{R}_y = E[\mathbf{y}_t \mathbf{y}_t^\top]$ and $\mathbf{r}_y = E[\mathbf{y}_t y(t)]$.

Using (7), (2) and Assumption A2, the autocovariance vector $\mathbf{r}_y$ is described as

$$\mathbf{r}_y = \mathbf{R}_y \mathbf{a} + \mathbf{r}_w - \mathbf{R}_w \mathbf{a} \tag{10}$$

where $\mathbf{R}_w = E[\mathbf{w}_t \mathbf{w}_t^\top]$ and $\mathbf{r}_w = E[\mathbf{w}_t w(t)]$. Combining (9) with (10) leads to

$$\mathbf{a}_{LS} = \mathbf{a} + \mathbf{R}_y^{-1}(\mathbf{r}_w - \mathbf{R}_w \mathbf{a}) \tag{11}$$

which gives an expression for the asymptotic bias of $\mathbf{a}_{LS}$:

$$\Delta\mathbf{a} \equiv \mathbf{a} - \mathbf{a}_{LS} = -\mathbf{R}_y^{-1}(\mathbf{r}_w - \mathbf{R}_w \mathbf{a}) \tag{12}$$

Note from (11) and (12) that $\mathbf{a}_{LS}$ is an asymptotically biased estimate if $\mathbf{R}_w \neq \mathbf{0}$ or $\mathbf{r}_w \neq \mathbf{0}$. In other words, the non-zero autocovariance functions $r_w(.)$ of the colored measurement noise $w(t)$ induce as well as determine the asymptotic bias $\Delta\mathbf{a}$.

3.2. Estimation of Noise Autocovariances

For convenience of illustration, we assume that $M = p$ in the remaining part of the paper, while the case of $M \neq p$ may be handled in a similar way without any substantial difficulties. With this assumption, the condition (3) becomes

$$r_w(k) = 0, \quad k = p, p+1, p+2, \ldots \tag{13}$$

So in order to implement the bias correction scheme, it suffices to estimate the autocovariance functions $r_w(0)$, $r_w(1)$, ..., $r_w(p-1)$, or the autocovariance vector $\mathbf{g}_w$ defined by

$$\mathbf{g}_w^\top = [r_w(0) \ r_w(1) \ldots r_w(p-1)] \tag{14}$$

For this purpose, we consider identifying the noisy AR system (7) using a model order of $2p$ instead of p. That is, we artificially rewrite the underlying p-th order noisy AR system (7) as a $2p$-th order model:

$$y(t) = \phi_t^\top \alpha + v(t) + w(t) - \omega_t^\top \alpha \tag{15}$$

where

$$\alpha^\top = [\mathbf{a}^\top; \ \bar{\mathbf{a}}^\top], \quad \bar{\mathbf{a}}^\top = [\bar{a}_{p+1} \ldots \bar{a}_{2p}] = \mathbf{0} \tag{16}$$

$$\phi_t^\top = [\mathbf{y}_t^\top; \ \bar{\mathbf{y}}_t^\top], \quad \bar{\mathbf{y}}_t^\top = [y(t-p-1) \ldots y(t-2p)] \tag{17}$$

$$\omega_t^\top = [\mathbf{w}_t^\top; \ \bar{\mathbf{w}}_t^\top], \quad \bar{\mathbf{w}}_t^\top = [w(t-p-1) \ldots w(t-2p)] \tag{18}$$

In particular, (16) shows that the p zero parameters $\bar{a}_{p+1}$, ..., $\bar{a}_{2p}$ are introduced into the identified noisy AR system.

Similarly to (9), the LS estimate of α is found to be

$$\alpha_{LS} = \mathcal{R}_y^{-1}\rho_y \tag{19}$$

where $\bar{\mathbf{R}}_y = E[\mathbf{y}_t\bar{\mathbf{y}}_t^\top]$, $\bar{\mathbf{r}}_y = E[\bar{\mathbf{y}}_t y(t)]$,

$$\mathcal{R}_y = E[\phi_t\phi_t^\top] = \begin{bmatrix} \mathbf{R}_y & \bar{\mathbf{R}}_y \\ \bar{\mathbf{R}}_y^\top & \mathbf{R}_y \end{bmatrix}, \quad \rho_y = E[\phi_t y(t)] = \begin{bmatrix} \mathbf{r}_y \\ \bar{\mathbf{r}}_y \end{bmatrix} \tag{20}$$

Moreover, the asymptotic expression for α_{LS} is given by

$$\alpha_{LS} = \alpha + \mathcal{R}_y^{-1}(\rho_w - \mathcal{R}_w\alpha) \tag{21}$$

where $\bar{\mathbf{R}}_w = E[\mathbf{w}_t\bar{\mathbf{w}}_t^\top]$, $\bar{\mathbf{r}}_w = E[\bar{\mathbf{w}}_t w(t)]$,

$$\mathcal{R}_w = E[\omega_t\omega_t^\top] = \begin{bmatrix} \mathbf{R}_w & \bar{\mathbf{R}}_w \\ \bar{\mathbf{R}}_w^\top & \mathbf{R}_w \end{bmatrix}, \quad \rho_w = E[\omega_t y(t)] = \begin{bmatrix} \mathbf{r}_w \\ \bar{\mathbf{r}}_w \end{bmatrix} \tag{22}$$

By the way of finding the LS estimate of the introduced parameter vector $\bar{\mathbf{a}}$ via (19) and (21), respectively, and by means of the condition (13), we can obtain

$$(\mathbf{R}_1\mathbf{Q}_1(\mathbf{a}) - \mathbf{Q}_2(\mathbf{a}) - \mathbf{R}_1\mathbf{T}_1)\mathbf{g}_w = \bar{\mathbf{r}}_y - \mathbf{R}_1\mathbf{r}_y \tag{23}$$

where

$$\mathbf{Q}_1(\mathbf{a}) = [(\mathbf{T}_0 + \mathbf{T}_0^\top)\mathbf{a} \;\; \ldots \;\; (\mathbf{T}_{p-1} + \mathbf{T}_{p-1}^\top)\mathbf{a}] \tag{24}$$

$$\mathbf{Q}_2(\mathbf{a}) = [\mathbf{T}_p\mathbf{a} \;\; \mathbf{T}_{p-1}\mathbf{a} \;\; \ldots \;\; \mathbf{T}_1\mathbf{a}] \tag{25}$$

$$\mathbf{R}_1 = \bar{\mathbf{R}}_y^\top\mathbf{R}_y^{-1} \tag{26}$$

$$\mathbf{T}_j = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{p-j} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathcal{R}^{p\times p}, \quad j = 1, ..., p-1 \tag{27}$$

$$\mathbf{T}_0 = \frac{1}{2}\mathbf{I}_p, \quad \mathbf{T}_p = \mathbf{0} \in \mathcal{R}^{p\times p} \tag{28}$$

The detailed derivation for (23) is omitted here due to limited space. Equation (23) is the key expression that provides a way for estimating the noise autocovariance vector $\mathbf{g}_w$.

3.3. ILS-CN Algorithm

On the basis of the above work, an ILS type algorithm is developed for identification of AR signals subject to Colored Noise, which is called the ILS-CN algorithm for short.

ILS-CN Algorithm

Step 1. Evaluate the autocovariance estimates $\hat{\mathbf{R}}_y$, $\hat{\bar{\mathbf{R}}}_y$, $\hat{\mathbf{r}}_y$ and $\hat{\bar{\mathbf{r}}}_y$ using the noisy measurements $\{y(t),\ 1 \le t \le N\}$, and let $\hat{\mathbf{R}}_1 = \hat{\bar{\mathbf{R}}}_y^\top\hat{\mathbf{R}}_y^{-1}$.

Step 2. Set the initial iteration estimate:

$$\hat{\mathbf{a}}_{ILS}^{(0)} \equiv \hat{\mathbf{a}}_{LS} = \hat{\mathbf{R}}_y^{-1}\hat{\mathbf{r}}_y \tag{29}$$

for $i = 0$, where the superscript i denotes the iteration step.

Step 3. Find the estimate of the measurement noise autocovariance vector $\mathbf{g}_w$:

$$(\hat{\mathbf{R}}_1\mathbf{Q}_1(\hat{\mathbf{a}}_{ILS}^{(i-1)}) - \mathbf{Q}_2(\hat{\mathbf{a}}_{ILS}^{(i-1)}) - \hat{\mathbf{R}}_1\mathbf{T}_1)\hat{\mathbf{g}}_w^{(i)} = \hat{\bar{\mathbf{r}}}_y - \hat{\mathbf{R}}_1\hat{\mathbf{r}}_y \tag{30}$$

and let

$$\hat{\mathbf{R}}_w^{(i)} = \sum_{j=0}^{p-1}(\mathbf{T}_j + \mathbf{T}_j^\top)r_w^{(i)}(j), \quad \hat{\mathbf{r}}_w^{(i)} = \mathbf{T}_1\hat{\mathbf{g}}_w^{(i)} \tag{31}$$

Step 4. Compute the estimate of the AR parameter vector $\mathbf{a}$ via the bias correction scheme:

$$\hat{\mathbf{a}}_{ILS}^{(i)} = \hat{\mathbf{a}}_{LS} - \hat{\mathbf{R}}_y^{-1}(\hat{\mathbf{r}}_w^{(i)} - \hat{\mathbf{R}}_w^{(i)}\hat{\mathbf{a}}_{ILS}^{(i-1)}) \tag{32}$$

Step 5. If convergence is achieved, terminate the iteration process; otherwise, repeat step 3.

3.4. Remarks

(i) When the measurement noise $w(t)$ is white, namely, $M = 1$ in the condition (3), the proposed algorithm reduces to the ILS method presented in [8], or the latter is just a special case of the ILS-CN algorithm. Thus, we have greatly extended the domain of application of the ILS based method so that it can handle the AR signal identification in the presence of colored noise.

(ii) Since $\hat{\mathbf{a}}_{LS}$ can be evaluated using the recursive LS procedure [2], the proposed ILS algorithm may be implemented recursively for on-line estimation. The relevant procedure has been derived.

(iii) In some signal processing applications, there is need to know the driving noise variance. To get an estimate of σ_v^2, we consider the average LS errors given by

$$J \equiv E[(y(t) - \mathbf{y}_t^\top\mathbf{a}_{LS})^2] = \sigma_v^2 + r_w(0) + \mathbf{a}_{LS}^\top(\mathbf{R}_w\mathbf{a} - \mathbf{r}_w) - \mathbf{r}_w^\top\mathbf{a} \tag{33}$$

The driving noise variance estimate is calculated as

$$\hat{\sigma}_v^2 = \hat{J} - \hat{r}_w^{(i)}(0) - \hat{\mathbf{a}}_{LS}^\top(\hat{\mathbf{R}}_w^{(i)}\hat{\mathbf{a}}_{ILS}^{(i)} - \hat{\mathbf{r}}_w^{(i)}) + \hat{\mathbf{r}}_w^{(i)\top}\hat{\mathbf{a}}_{ILS}^{(i)} \tag{34}$$

where $\hat{J} = \frac{1}{N}\sum_{t=1}^{N}[y(t) - \mathbf{y}_t^\top\hat{\mathbf{a}}_{LS}]^2$.

(iv) The convergence of the ILS-CN algorithm can be analyzed in a similar fashion to the ILS type methods presented in [7] and [8]. Since it is developed based on the bias correction principle, the ILS-CN algorithm is well motivated and can produce the unbiased estimate of the AR parameters. This is verified by the simulation results given in the next section. Besides, the fact that the ILS-CN algorithm is built on linear regression assures that it does not involve any intensive computations. This is another attractive aspect of the ILS-CN algorithm.

(v) Although the MYW method may be extendible to the case where the observation noise is an MA noise, the necessity of use of high order Yule-Walker equations may further impair its estimation accuracy as well as numerical efficiency.

4. SIMULATION RESULTS

Computer simulations of the proposed ILS-CN algorithm have been conducted. The simulated noisy AR system is described by

$$A(q^{-1}) = 1 - 1.9q^{-1} + 1.3q^{-2} - 0.28q^{-3} \tag{35}$$

$$\sigma_v^2 = 1.0, \quad w(t) = (1 - 1.0q^{-1} + 0.2q^{-2})e(t) \tag{36}$$

where $\{e(t)\}$ is a zero-mean white noise sequence with variance $\sigma_e^2 = 0.1$. The signal-to-noise ratio (SNR) is defined as

$$\text{SNR} = 10\log_{10}\frac{E[x(t)^2]}{E[w(t)^2]} \text{ dB} \tag{37}$$

which gives SNR=20dB in this example. The proposed ILS-CN algorithm was applied for off-line identification, while the termination criterion was selected as whenever the relative error between two consecutive AR parameter estimates is less than 0.001. Table 1 displays the arithmetic means and standard deviations of the estimated a_1, a_2, a_3, σ_v^2, $r_w(0)$, $r_w(1)$ and $r_w(2)$ based on 100 independent tests of $N = 2000$ data points. The results by the standard LS and the MYW methods are also included in the table for comparison. As expected, the ILS-CN algorithm produces the estimates with desirable accuracy in the presence of colored noise. Also, the proposed algorithm is less computationally demanding. We notice that the standard LS estimates are seriously biased while the MYW method fails to work properly.

5. SUMMARY

The primary contribution of this paper is that some significant extensions on the recently proposed ILS method have been made, with a view to identifying AR signals subject to colored noise. A useful expression has been derived for estimation of the autocovariance functions of the colored measurement noise. This is the key to implementing the bias correction scheme so as to achieve the unbiased estimate of the AR parameters. The developed ILS-CN method can be used for both off-line and on-line estimation of AR signals in the presence of colored noise. Results of Monte-Carlo simulations verify the theoretical analysis.

6. REFERENCES

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Table 1. Simulation Results
(RE = $\|\hat{\mathbf{a}} - \mathbf{a}\|/\|\mathbf{a}\|$, NFPT = No. of flops per test, NIPT = No. of iterations per test)

method	a_1	a_2	a_3	σ_v^2	$r_w(0)$	$r_w(1)$	$r_w(2)$	RE	NFPT	NIPT
LS	1.2682 ±0.0228	−0.2014 ±0.0364	−0.2876 ±0.0226	—	—	—	—	59.87%	36163	—
MYW	0.6774 ±5.7771	0.5654 ±8.6402	−0.5967 ±3.9740	—	—	—	—	103.34%	84180	—
ILS-CN	1.8393 ±0.5877	−1.2048 ±0.4152	0.2358 ±0.2840	0.9586 ±0.8259	0.2267 ±0.5954	−0.0820 ±0.2679	0.0087 ±0.1056	5.23%	105140	7.1
true value	1.9	−1.3	0.28	1.0	0.204	−0.12	0.02			