MEAN WEIGHT BEHAVIOR OF THE FILTERED-X LMS ALGORITHM

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ABSTRACT

This paper presents a stochastic analysis of the Filtered-X LMS algorithm. The mean weight vector recursion is derived for slow adaptation and for a white reference signal without use of independence theory. The Wiener solution is determined explicitly as a function of the input statistics and the impulse responses of the primary and secondary signal paths. It is shown that the steady-state mean weights for the Filtered-X LMS algorithm converge to the Wiener solution only if the estimate of the secondary path is without error. Monte Carlo simulations show excellent agreement with the behavior predicted by the theoretical model.

1. **INTRODUCTION**

Acoustic noise control has become ever more important in recent years. National and multinational programs and policies are being established to reduce and control environmental noise. Some of the most important breakthroughs in control of sound and vibration have been achieved through the use of feedforward active noise control techniques. Active control system performance is largely determined by the physical implementation and by the proper choice and design of the adaptive algorithm.

The most popular adaptive algorithm for active noise and vibration control is the Filtered-X LMS Algorithm [1,2]. This algorithm is a modification of the well known LMS algorithm. Here, the reference signal is filtered to compensate for a filtering operation which is inherent in the acoustic adaptation loop (i.e. speakers, microphones and other acoustic transducers). The introduction of these two filters in the system significantly complicates the analysis of the adaptive algorithm behavior. Analysis results, derived for the conventional LMS algorithm, do not apply to the filtered case. Also, simplifying assumptions for LMS algorithm analysis cannot be easily extended to the Filtered-X LMS algorithm. This comment applies to the socalled independence theory (i.e. successive data vectors are assumed statistically independent). Data vector correlations are created by the loop filtering operations. These correlations render the independence theory model inadequate for the statistical analysis of the algorithm. This is a substantial theoretical drawback. Exact analysis of the algorithm is very cumbersome without the independence assumption even for the conventional LMS algorithm [3]. Most of the stochastic analyses of the Filtered-X LMS algorithm in the literature concentrate on

algorithm stability [1,2,4,5], which is important for the proper algorithm design. However, more complete analytical models are necessary for predicting the algorithm's transient and steady-state behavior under different implementation conditions.

Recently, some results have been presented on the stochastic analysis of the Filtered-X algorithm. A recursion has been derived in [4] for the mean weight behavior. The analysis uses the independence theory and assumes extremely slow convergence. A more extensive analysis is presented in [6]. However, this analysis is also based on the independence assumption. Detailed results are derived for the Delayed LMS algorithm, a special case of Filtered-X LMS. Determining the model parameters requires the experimental measurement of the stability bounds of the algorithm.

This paper presents a stochastic analysis of the Filtered-X LMS algorithm. A vector recursion is obtained for the mean weight for slow adaptation and a white reference without using the independence assumption. The Wiener solution is determined explicitly as a function of the input statistics and the response of the primary and secondary signal paths. It is shown that the steady-state mean weights for Filtered-X LMS converge to the Wiener solution if the estimate of the secondary path is without error. Monte Carlo simulations show excellent agreement with the theoretical predictions.

2. **ANALYSIS**

2.1 The Analysis Model

Figure 1 shows a block diagram of an active noise control problem which uses the Filtered-X LMS algorithm. The notation

- $\mathbf{W}^{\mathbf{0}} = \begin{bmatrix} w_0^{\mathbf{0}}, w_1^{\mathbf{0}}, \cdots, w_{N-1}^{\mathbf{0}} \end{bmatrix}^T$: impulse response to be identified $\mathbf{W}_n = \begin{bmatrix} w_{0_n}, w_{1_n}, \cdots, w_{N-1_n} \end{bmatrix}^T$: adaptive weight vector $\mathbf{S} = \begin{bmatrix} s_0 & s_1 & \cdots & s_{M-1} \end{bmatrix}^T$: impulse response of the system in the auxiliary path $\hat{\mathbf{S}} = \left[\hat{s}_0 \ \hat{s}_1 \cdots \hat{s}_{\hat{M}-1}\right]^T : \text{ estimate of the impulse response of the}$
- system in the auxiliary path

$$\begin{split} \mathbf{X}_{n} &= \begin{bmatrix} x_{n}, x_{n-1}, \cdots, x_{n-N+1} \end{bmatrix}^{T} : \text{ observed data vector (reference signal)} \\ \mathbf{X}_{f_{n}} &= \begin{bmatrix} x_{f_{n}}, x_{f_{n-1}}, \cdots, x_{f_{n-N+1}} \end{bmatrix}^{T} : \text{ filtered reference signal} \end{split}$$

 d_n : primary signal

 z_n : white noise (uncorrelated with any other signal) with variance σ_z^2 .



Figure 1. Block diagram - active noise control system.

For the analysis, x_n is assumed white with variance σ_x^2 . Also, the dimensions of $\mathbf{W}^{\mathbf{0}}$ and \mathbf{W}_n are assumed the same for notational simplicity. If one vector has more components than the other, the additional components can be assumed to be zero.

2.2 Weight Vector Update Equation

The signals in Fig. 1 can be described by the expressions

$$e_n = d_n - y_{ln} + z_n \tag{1}$$

$$d_n = \mathbf{X}_n^T \mathbf{W}^\mathbf{0} \tag{2}$$

$$y_n = \mathbf{X}_n^T \mathbf{W}_n = \mathbf{W}_n^T \mathbf{X}_n \tag{3}$$

$$y_{ln} = \sum_{i=0}^{M-1} s_i \ y_{n-i} = \sum_{i=0}^{M-1} s_i \ \mathbf{X}_{n-i}^T \mathbf{W}_{n-i} , \qquad (4)$$

$$x_{f_n} = \sum_{i=0}^{\hat{M}-1} \hat{s}_i \, x_{n-i} \tag{5}$$

$$\mathbf{X}_{f_n} = \sum_{i=0}^{\hat{M}-1} \hat{s}_i \, \mathbf{X}_{n-i} \tag{6}$$

Substituting (2)-(4) in (1) yields

$$e_n = \mathbf{X}_n^T \mathbf{W}^{\mathbf{o}} - \sum_{i=0}^{M-1} s_i \, \mathbf{X}_{n-i}^T \mathbf{W}_{n-i} + z_n \tag{7}$$

The weight update equation for the Filtered-X LMS algorithm is given by

$$\mathbf{W}_{n+1} = \mathbf{W}_n + \mu e_n \, \mathbf{X}_{f_n} \tag{8}$$

Substituting (6) and (7) into (8) yields

$$\mathbf{W}_{n+1} = \mathbf{W}_n + \mu \left(\sum_{j=0}^{\hat{M}-1} \hat{s}_j \mathbf{X}_{n-j} \mathbf{X}_n^T \mathbf{W}^{\mathbf{0}} - \sum_{i=0}^{M-1} \sum_{j=0}^{\hat{M}-1} s_i \hat{s}_j \mathbf{X}_{n-j} \mathbf{X}_{n-i}^T \mathbf{W}_{n-i} + \sum_{j=0}^{\hat{M}-1} \hat{s}_j \mathbf{X}_{n-j} z_n \right)$$
(9)

2.3 Mean Weight Vector Behavior

Taking the expected value of (9) yields

$$E[\mathbf{W}_{n+1}] = E[\mathbf{W}_n] + \mu \left(\sum_{j=0}^{\hat{M}-1} \hat{s}_j E[\mathbf{X}_{n-j} \mathbf{X}_n^T] \mathbf{W}^{\mathbf{o}} - \sum_{i=0}^{M-1} \sum_{j=0}^{\hat{M}-1} s_i \hat{s}_j E[\mathbf{X}_{n-j} \mathbf{X}_{n-i}^T \mathbf{W}_{n-i}] + \sum_{j=0}^{\hat{M}-1} \hat{s}_j E[\mathbf{X}_{n-j} z_n] \right)$$
(10)

The expectations in the summations on the right side of (10) are now evaluated. Using the notation

$$R_{j-i} = E\left[\mathbf{X}_{n-j}\mathbf{X}_{n-i}^{T}\right]$$
(11)

for the correlation matrix of the reference vectors, the (k, ℓ) th entry of R_{j-i} is given by

$$\begin{bmatrix} \mathbf{R}_{j-i} \end{bmatrix}_{k\,\ell} = E \begin{bmatrix} x_{n-j-k+1} x_{n-i-\ell+1} \end{bmatrix}$$
$$= \begin{cases} \sigma_x^2 & \text{for } j-i = \ell - k \\ 0 & \text{otherwise} \end{cases}$$
(12)

The expectations in the first summation are then equal to R_j , $j = 0, ..., \hat{M} - 1$.

For small μ , the correlations between \mathbf{W}_{n-i} and either \mathbf{X}_{n-j} or \mathbf{X}_{n-i} can be disregarded in determining the expectations in the second summation. However, the correlations between the input vector pairs must be calculated because of \mathbf{S} . With the above approximation, the expectations in the second summation become

$$E\left[\mathbf{X}_{n-j}\mathbf{X}_{n-i}^{T}\mathbf{W}_{n-i}\right] \approx E\left[\mathbf{X}_{n-j}\mathbf{X}_{n-i}^{T}\right]E\left[\mathbf{W}_{n-i}\right]$$

= $\mathbf{R}_{j-i}E\left[\mathbf{W}_{n-i}\right]$ (13)

The expectations in the third term are equal to zero because z_n is zero mean and uncorrelated with x_n . Thus, (10) can be written as

$$E\left[\mathbf{W}_{n+1}\right] = E\left[\mathbf{W}_{n}\right] - \mu\left(\sum_{i=0}^{M-1}\sum_{j=0}^{\hat{M}-1}s_{i}\hat{s}_{j}\mathbf{R}_{j-i}E\left[\mathbf{W}_{n-i}\right] - \sum_{j=0}^{\hat{M}-1}\hat{s}_{j}\mathbf{R}_{j}\mathbf{W}^{o}\right) (14)$$

It is clear from (14) that the weight vector converges to

$$E[\mathbf{W}_{\infty}] = \left[\sum_{i=0}^{M-1} \sum_{j=0}^{\hat{M}-1} s_i \hat{s}_j \mathbf{R}_{j-i}\right]^{-1} \left[\sum_{j=0}^{\hat{M}-1} \hat{s}_j \mathbf{R}_j \mathbf{W}^{\mathbf{0}}\right]$$
(15)

Compare (14) with (3) in [4]. The latter assumes that $E[\mathbf{W}_{n-i}]$ is constant for all $i = 0, \dots, M-1$. This assumption corresponds to extremely slow adaptation which is not used here. Equation (14) can be used to study the algorithm behavior for different estimates $\hat{\mathbf{S}}$ of the secondary path response \mathbf{S} .

3. WIENER SOLUTION

A sufficient order conventional LMS algorithm, applied to system identification, corresponds to an unconstrained optimization problem. The optimum solution is obtained by matching the impulse response of the adaptive filter and the unknown system. The adaptive filter response is convolved with the impulse response of the secondary path filter **S** for the Filtered-X LMS algorithm. Thus, a linear combination of the adaptive weights should be equal to each sample of the unknown system response. This is a constrained optimization problem, frequently denoted constrained adaptive filtering. Therefore, it is important to determine 1) the optimum weight vector, 2) the secondary path filter impulse response and 3) the Wiener solution and its relationship to the W^0 of the physical path.

For the Wiener solution for the Filtered-X LMS algorithm, consider Fig. 2.



Figure 2. Block diagram of the constrained adaptive filter used to determine the Wiener solution

The mean-square error for a constant weight vector can be obtained from (1) and (4) as

$$E\left[e_{n}^{2}\right] = E\left[d_{n}^{2}\right] - 2\left[\sum_{i=0}^{M-1} s_{i} \mathbf{P}_{i}^{T}\right] \mathbf{W} + \mathbf{W}^{T}\left[\sum_{i=0}^{M-1} \sum_{j=0}^{M-1} s_{j} s_{j} \mathbf{R}_{i-j}\right] \mathbf{W} \quad (16)$$

where $\mathbf{P}_{j-i} = E[d_{n-j} \mathbf{X}_{n-i}]$ is the cross-correlation vector between the primary and reference signals. It is easy to show

from (16) that the Wiener solution (minimum $E[e_n^2]$) is given by

$$\mathbf{W}_{w} = \widetilde{\mathbf{R}}^{-1} \ \widetilde{\mathbf{P}}$$
(17)

where
$$\widetilde{\mathbf{R}} = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} s_i s_j \mathbf{R}_{i-j}$$
 and $\widetilde{\mathbf{P}} = \sum_{i=0}^{M-1} s_i \mathbf{P}_i^T$.

Using the definition of \mathbf{P}_i and the expression $d_n = \mathbf{X}_n^T \mathbf{W}^{\mathbf{o}}$ in (17) yields

$$\mathbf{W}_{w} = \widetilde{\mathbf{R}}^{-1} \left[\sum_{i=0}^{M-1} s_{i} \mathbf{R}_{i} \right] \mathbf{W}^{\mathbf{0}}$$
(18)

Equation (18) is the Wiener solution, expressed as a function of the input statistics and the acoustic path response for the system in Fig. 1. As expected, note that the optimum solution is not $\mathbf{W}^{\mathbf{0}}$. More importantly, the Wiener solution (the minimum that can be achieved by the algorithm) may be quite different than the weight vector with the first *N* samples of $\mathbf{W}^*\mathbf{S}$ (convolution of \mathbf{W} and \mathbf{S}) matching the corresponding entries of $\mathbf{W}^{\mathbf{0}}$. Comparison of (18) and (15) shows that $\hat{\mathbf{S}} = \mathbf{S}$ leads to $E[\mathbf{W}_{\infty}] = \mathbf{W}_{w}$. Moreover, (18) can be used to determine the weight vector misadjustment for imperfect estimation of \mathbf{S} .

Substituting \mathbf{W}_{w} for \mathbf{W} in (16) yields the minimum meansquare error for the Filtered-X LMS algorithm:

$$\min\left\{E\left[e_{n}^{2}\right]\right\}=E\left[z_{n}^{2}\right]+\mathbf{W}^{\mathbf{o}^{T}}\left\{\mathbf{R}_{0}-\left(\sum_{i=0}^{M-1}s_{i}\mathbf{R}_{i}^{T}\right)\widetilde{\mathbf{R}}^{-1}\left(\sum_{i=0}^{M-1}s_{i}\mathbf{R}_{i}\right)\right\}\mathbf{W}^{\mathbf{o}}$$
 (19)

4. SIMULATIONS

This section presents some simulation examples which demonstrate the accuracy of the analytical model (14). Two plots are shown for each example: The weight convergence behavior (how well the model (14) predicts the weight behavior), and the identification error

$$\boldsymbol{\varepsilon}_{n} = \left\| \mathbf{W}^{\mathbf{o}} \right\| - \left\| \mathbf{W}_{n} * \mathbf{S} \right\| \tag{20}$$

which measures the identification accuracy achievable with the Filtered-X LMS algorithm. In all the examples, $\sigma_x^2 = 1$ and $\sigma_z^2 = 0.01$. Figures 3-6 show the simulation results and the theoretical predictions obtained using (14). The ragged curves correspond to averages of 20 runs.

4.1 Example 1

Perfect estimation of the secondary path response ($\mathbf{S}=\hat{\mathbf{S}}$):

 $\mathbf{W}^{\mathbf{0}} = [1.5, 1.0, -0.7], \mathbf{S} = \hat{\mathbf{S}} = [0.2, 0.8], \mu = 0.002$



Figure 3. Evolution of ε_n for Example 1



Figure 4. Evolution of \mathbf{W}_n for Example 1

Figures 3 and 4 show excellent agreement between theory and simulation. The weights converge to the Wiener solution (18). Figure 3 clearly shows the distance between the optimum solution and the matching between W*S and W° .

4.2 Example 2

Different filters in **S** and $\hat{\mathbf{S}}$ (imperfect estimation)

$$\mathbf{W}^{\mathbf{o}} = \begin{bmatrix} 1.5, 1.0, -0.7 \end{bmatrix}^{T}, \ \mathbf{S} = \begin{bmatrix} 0.2, 0.8 \end{bmatrix}^{T}, \ \mathbf{\hat{S}} = \begin{bmatrix} 0.4, 0.9 \end{bmatrix}^{T}, \\ \mu = 0.005$$

The results shown in Figures 5 and 6 show excellent agreement between theory and simulation for the imperfect estimation case. The Wiener solution has not been achieved because of the inaccurate estimate of S.



Figure 5. Evolution of ε_n for Example 2



Figure 6. Evolution of \mathbf{W}_n for Example 2

5. CONCLUSIONS

Accurate results for the stochastic analysis of the Filtered-X LMS algorithm cannot rely on the independence assumption. An exact analysis requires consideration of all existing signal correlations. This represents an enormous mathematical task. The question is then what simplifications can be made which lead to a tractable mathematical problem and to accurate models. When determining the weight vector behavior for slow learning, a number of authors have demonstrated the following: the correlation over time between reference signal vectors is much more important than the correlation between the weight vector and the reference signal vector. This behavior is due to the tapped delay line structure of the algorithm which is often ignored (with success) in much of the analysis. Here, this timecorrelation approach leads to an accurate model of the Filtered-X LMS statistical algorithm behavior, which is not the case using the independence assumption.

6. **REFERENCES**

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