IDENTIFICATION OF CLOSED-LOOP LINEAR SYSTEMS VIA CYCLIC SPECTRAL ANALYSIS: AN EQUATION-ERROR FORMULATION

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ABSTRACT

The problem of closed-loop system identification given noisy input-output measurements is considered. The closed-loop system operates under an external cyclostationary input which is not measured. Noisy measurements of the (direct) input and output of the plant are assumed to be available. The various disturbances affecting the system are either stationary or cyclostationary with cycle frequencies different from the input cycle frequencies. The closed-loop system must be stable but it is allowed to be unstable in open-loop. A frequency-domain parametric solution is proposed and analyzed using an equation error formulation, and cyclic spectrum and cross-spectrum of the input-output measurements. The parameter estimator is shown to be consistent. A simulation example using an unstable open-loop system is presented to illustrate the proposed approach.

1. INTRODUCTION

Consider the following widely used input-output linear system model:

$$y(t) \;=\; H(q)u(t) \,+\, e(t) \qquad (1-1)$$

where $\{u(t)\}$ is the measured input sequence, t is discrete-time, $\{y(t)\}$ is the noisy output, and $\{e(t)\}$ is a measurement noise (disturbance) sequence. With q^{-1} denoting the backward-shift operator (i.e. $q^{-1}u(t) = u(t-1)$), the linear system $H(q)$ represents an IIR system:

$$H(q) \;=\; \sum_{i=1}^{\infty} h(i)q^{-i}. \qquad (1-2)$$

The above system operates in a closed-loop (see Fig. 1) where the input $u(t)$ in (1-1) is determined through linear feedback as

$$u(t) \;=\; v(t) \,-\, F(q)y(t) \qquad (1-3)$$

where $F(q) = \sum_{i=0}^{\infty} f(i)q^{-i}$ is the controller transfer function and $\{v(t)\}$ is some external input sequence.

Given an input-output record $\{u(t), y(t), \;\; t = 1, 2, \cdots\}$, but the underlying true system model $H(q)$ *unknown*, it is of much interest in control, communications and signal processing applications to fit a rational function model

$$G(q) \;:=\; \frac{B(q)}{A(q)} \;=\; \frac{\sum_{i=1}^{n_b} b_i q^{-i}}{1 \,+\, \sum_{i=1}^{n_a} a_i q^{-i}} \qquad (1-4)$$

to given input-output record [1]-[5]. A wide variety of approaches exist [1]-[5],[8],[9].

The main objective of this paper is to provide a frequency-domain solution using cyclic spectral analysis to the problem of closed-loop system identification given time-domain input-output data. In the presence of the feedback and noise $e(t)$, the input $\{u(t)\}$ is correlated with the output $\{y(t)\}$ so that the standard spectral analysis (and related approaches) yields biased estimators of the system transfer function and related parameters. For further details, see [1], [4, Chapter 10] and [5]. Recently in [1] a nonparametric approach using cyclostationary and/or non-Gaussian inputs was presented to solve this problem. Ref. [1] requires the open-loop transfer function to be stable and the approach presented therein is nonparametric. In this paper we focus on second-order cyclostationarity and parametric approaches. *Unlike [1], we allow the open-loop transfer function $H(e^{j\omega})$ to be unstable.*

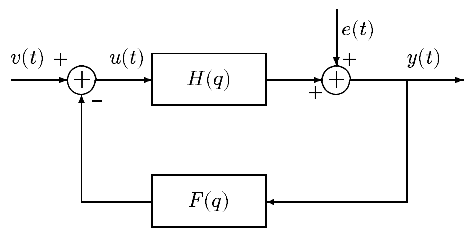

Fig. 1. Closed-loop system block diagram.

2. PRELIMINARIES

A zero-mean discrete random sequence $\{u(t)\}$ with its second-order cumulant function $c_{uu}(t;\tau) = \mathrm{cum}\{u(t+\tau), u(t)\} = E\{u(t+\tau)u(t)\}$ is called a second-order almost cyclostationary sequence if its second-order cumulant function is an almost periodic function in t [7]. Assume that $c_{uu}(t;\tau)$ admits a Fourier series representation w.r.t. t. Then

$$c_{uu}(t;\tau) = \sum_{\alpha \in A_{uu}} C_{uu}(\alpha;\tau)e^{j\alpha t}, \qquad (2-1)$$

$$C_{uu}(\alpha;\tau) = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} c_{uu}(t;\tau)e^{-j\alpha t}, \qquad (2-2)$$

$$A_{uu} := \{\alpha : C_{uu}(\alpha;\tau) \neq 0, \;\, 0 \leq \alpha < 2\pi\}. \qquad (2-3)$$

The Fourier coefficient $C_{uu}(\alpha;\tau)$ is called the second-order cyclic cumulant at cycle frequency α [7]. The set A_{uu} is the countable set of cycle frequencies of the second-order cyclic cumulant of $\{u(t)\}$. The time-varying and cyclic cumulant spectra, respectively, of $\{u(t)\}$ are defined as

$$\mathrm{S}_{uu}(t;\omega) := \sum_{\tau=-\infty}^{\infty} c_{uu}(t;\tau)e^{-j\omega\tau}, \qquad (2-4)$$

$$S_{uu}(\alpha;\omega) := \sum_{\tau=-\infty}^{\infty} C_{uu}(\alpha;\tau)e^{-j\omega\tau}. \qquad (2-5)$$

Suppose that $u(t)$ is filtered by a linear, time-invariant, BIBO stable filter $G(q) = \sum_i g(i)q^{-i}$. Consider the second-order cross-cumulant function

$$c_{yu}(t;\tau) := \mathrm{cum}\{y(t+\tau), u(t)\} = E\{y(t+\tau)u(t)\}$$

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$$= \sum_{m=-\infty}^{\infty} g(m) c_{uu}(t;\tau - m). \qquad (2-6)$$

Mimicking (2-2), define the cyclic cross-cumulant

$$C_{yu}(\alpha;\tau) := \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} c_{yu}(t;\tau) e^{-j\alpha t}. \qquad (2-7)$$

Using (2-2), (2-6) and (2-7), we have

$$C_{yu}(\alpha;\tau) = \sum_{m=-\infty}^{\infty} C_{uu}(\alpha;\tau - m) g(m). \qquad (2-8)$$

Mimicking (2-5), define the cyclic cross-spectrum

$$S_{yu}(\alpha;\omega) := \sum_{\tau=-\infty}^{\infty} C_{yu}(\alpha;\tau) e^{-j\omega\tau} = S_{uu}(\alpha;\omega) G(e^{j\omega}) \qquad (2-9)$$

where we have used (2-5) and (2-8), and

$$G(e^{j\omega}) := G(q)\big|_{q=e^{j\omega}} = \sum_m g(m) e^{-j\omega m}. \qquad (2-10)$$

Finally, if $c_{uy}(t;\tau) = E\{u(t+\tau)y(t)\}$ etc., then it follows in a manner similar to (2-6)-(2-10) that

$$S_{uy}(\alpha;\omega) = S_{uu}(\alpha;\omega) G(e^{j(\alpha-\omega)}) \qquad (2-11)$$

3. MODEL ASSUMPTIONS

We impose the following conditions on (1-1):

(AS1) $H(q)$ is strictly causal, that is $\lim_{q\to\infty} H(q) = 0$, so that $y(t)$ depends only on past input values.

(AS2) $[1 + H(q)F(q)]^{-1}$ is asymptotically stable.

(AS3) $\{e(t)\}$ and $\{v(t)\}$ are zero-mean almost cyclostationary random processes with cycle frequency sets A_{ee} and A_{vv}, respectively. There exists a known set of non-zero cycle frequencies $A_{\mathcal{M}} = \{\alpha\} \subset A_{vv}$ for which $\mathcal{S}_{ev}(\alpha;\omega) = \mathcal{S}_{ee}(\alpha;\omega) = 0$ and $\mathcal{S}_{vv}(\alpha;\omega) \neq 0$.

(AS4) For some $\alpha \in A_{\mathcal{M}}$, $|\mathcal{S}_{vv}(\alpha;\omega)| > 0$ for almost all $\omega \in [0,\pi]$ if the proposed approaches utilize the entire frequency range $[0,\pi]$. If only finite number of frequencies are used then $|\mathcal{S}_{vv}(\alpha;\omega)|$ need be non-zero only for this frequency set.

(AS5) Let $\{z_m(t)\}_{m=0}^{k}$ be $(k+1)$ random sequences such that $z_m(t) \in \{y(t), u(t), v(t), e(t)\}$. Let $\underline{\tau}_{(k)} := [\tau_1, \ldots, \tau_k]^T$. Let $c_{\mathbf{Z}}(t;\underline{\tau}_{(k)})$ denote the $(k+1)$st-order joint cumulant function $\mathrm{cum}\{z_0(t), z_1(t+\tau_1), \ldots, z_k(t+\tau_k)\}$. Let

$$C_{\mathbf{Z}}(\alpha;\underline{\tau}_{(k)}) := \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} c_{\mathbf{Z}}(t;\underline{\tau}_{(k)}) e^{-j\alpha t}.$$

The following summability conditions hold true for each $j = 1, \ldots, k$ and each $k = 1, 2, 3, \cdots$:

$$\sum_{\tau_1,\ldots,\tau_k=-\infty}^{\infty} \sup_t [1 + |\tau_j|] |c_{\mathbf{Z}}(t;\underline{\tau}_{(k)})| < \infty \quad (3\text{-}1)$$

$$\sum_{\tau_1,\ldots,\tau_k=-\infty}^{\infty} \sup_t [1 + |\tau_j|] |C_{\mathbf{Z}}(\alpha;\underline{\tau}_{(k)})| < \infty \quad (3\text{-}2)$$

for each $\alpha \in A_{\mathcal{M}}$.

Let the vector of unknown parameter be given by

$$\theta = [a_1 \ \cdots \ a_{n_a} \ b_1 \ \cdots \ b_{n_b}]^T. \qquad (3-3)$$

4. A FREQUENCY-DOMAIN SOLUTION

It follows from (1-1)-(1-3) that

$$y(t) = Q(q)v(t) + \eta(t) = s(t) + \eta(t), \qquad (4-1)$$

where the closed-loop transfer function $Q(q)$ and the output error sequence $\eta(t)$ are given by

$$Q(q) = H(q)[1 + H(q)F(q)]^{-1}, \qquad (4\text{-}2)$$
$$\eta(t) = [1 + H(q)F(q)]^{-1} e(t). \qquad (4\text{-}3)$$

From (1-1) and (1-3), under (AS2), we have

$$u(t) = \frac{1}{1 + H(q)F(q)} [v(t) - F(q)e(t)]. \qquad (4-4)$$

Using (2-6)–(2-11), (4-1)–(4-4) and (AS3), we have

$$S_{uu}(\alpha;\omega) = D^{-1}(\alpha;e^{j\omega}) S_{vv}(\alpha_l;\omega), \qquad (4\text{-}5)$$
$$S_{yu}(\alpha;\omega) = H(e^{j\omega}) D^{-1}(\alpha;e^{j\omega}) S_{vv}(\alpha;\omega), \quad (4\text{-}6)$$

for each $\alpha \in A_{\mathcal{M}}$ where

$$D(\alpha;e^{j\omega}) = [1 + H(e^{j\omega})F(e^{j\omega})][1 + H(e^{j(\alpha-\omega)})F(e^{j(\alpha-\omega)})]. \qquad (4-7)$$

It follows from (4-5) and (4-6) that under (AS2)-(AS4),

$$H(e^{j\omega}) = \frac{S_{yu}(\alpha;\omega)}{S_{uu}(\alpha;\omega)} \quad \forall \alpha \in A_{\mathcal{M}}. \qquad (4-8)$$

The basic approach to model parameter estimation consists of two steps. First obtain a consistent estimator $\widehat{H}(e^{j\omega})$ of $H(e^{j\omega})$ via consistent estimators $\widehat{S}_{yu}(\alpha;\omega)$ and $\widehat{S}_{uu}(\alpha;\omega)$ of $S_{yu}(\alpha;\omega)$ and $S_{uu}(\alpha;\omega)$, respectively, based upon the input-output record $\{u(t), y(t),\ t = 1, 2, \cdots, T\}$. Next estimate the system parameters using the estimated transfer function matrix as "data;" this part follows [3]. For simplicity, we consider estimates based on only a single cyclic frequency $\alpha \in A_{\mathcal{M}}$.

4.1. Transfer Function Estimator

This requires estimation of $S_{yu}(\alpha;\omega)$ and $S_{uu}(\alpha;\omega)$ for some known α in $A_{\mathcal{M}}$ (see (AS3)). We will follow the approach of smoothing in frequency domain [7]. Given a record of length T, let $Y(\omega)$ denote the DFT of $\{y(t),\ 1 \leq t \leq T\}$ given by

$$Y(\omega_k) = \sum_{t=0}^{T-1} y(t+1) \exp(-j\omega_k t), \qquad (4-9)$$

$$\omega_k = \frac{2\pi}{T} k, \quad k = 0, 1, \cdots, T-1. \qquad (4-10)$$

Similarly define $U(\omega_k)$. Given the above DFT's, following [7] we define the cross- and auto-cyclic-spectrum estimators as

$$\widehat{S}_{yu}(\alpha;k) = \frac{2\pi}{T^2} \sum_{s=1}^{T-1} W^{(T)}(\omega_{k-s}) Y(\omega_s) U(\alpha - \omega_s), \ (4-11)$$

$$\widehat{S}_{uu}(\alpha;k) = \frac{2\pi}{T^2}\sum_{s=1}^{T-1} W^{(T)}(\omega_{k-s})U(\omega_s)U(\alpha-\omega_s) \quad (4-12)$$

for $1 \leq k \leq T-1$, where the scalar weighting function $W^{(T)}(\alpha)$ is given by

$$W^{(T)}(\alpha) \ = \ B_T^{-1}\sum_{i=-\infty}^{\infty} W\left(B_T^{-1}[\alpha+2\pi i]\right) \qquad (4-13)$$

such that $W(\beta)$, $-\infty < \beta < \infty$, is real-valued, even, of bounded variation satisfying $\int_{-\infty}^{\infty} W(\beta)\,d\beta \ = \ 1$ and $\int_{-\infty}^{\infty} |W(\beta)|\,d\beta < \infty$ [6, Secs. 5.6 and 7.4]. It is convenient to take $W(\beta) = 0$ for $|\beta| > 2\pi$ and $W(\beta) = (4\pi)^{-1}$ for $|\beta| \leq 2\pi$. In this case (4-11) involves uniform weighting of the $2B_TT+1$ cyclic cross-periodogram ordinates whose frequencies fall in the interval $(\omega_k - 2\pi B_T, \omega_k + 2\pi B_T)$. Thus (4-11) reduces to

$$\widehat{S}_{yu}(\alpha;k) = \frac{1}{T(2m_T+1)}\sum_{i=-m_T}^{m_T} Y(\omega_{k-i})U(\alpha-\omega_{k-i}), \qquad (4-14)$$

where $m_T = B_TT$. Similar modification holds for (4-12).

Lemma 1. Let B_T be such that as $T \to \infty$, we have $B_T \to 0$ and $B_TT \to \infty$. Let $k_l(T)$ with $T = 1, 2, \cdots$ be a sequence of integers such that $\lim_{T\to\infty} k_l(T)/T \ = \ f_l$, a fixed frequency (in Hz). Then under (AS5),

$$\lim{}_{T\to\infty} E\{\widehat{S}_{yu}(\alpha;k_l(T))\} = S_{yu}(\alpha;2\pi f_l), \qquad (4-15)$$

$$\text{var}\left(\widehat{S}_{yu}(\alpha;k_l(T))\right) \ = \ O(\Delta_T^{-1}) \qquad (4-16)$$

where $\text{var}(X) := E\{|X|^2\} - |E\{X\}|^2$ and

$$\Delta_T \ = \ \frac{B_TT}{2\pi\int_{-\infty}^{\infty} W^2(\beta)d\beta}, \qquad (4\text{-}17)$$

$$= \ 2m_T+1, \ \text{ if (4-14) is used.} \qquad (4\text{-}18)$$

Convergence in (4-15)-(4-16) is uniform in f. •

Proof: It follows from some straightforward modifications of the proof of Theorem 3.3 in [7]. Theorem 3.3 of [7] deals with cyclic auto-spectrum whereas here we are also concerned with cyclic cross-spectrum. □

Clearly, Lemma 1 holds true when we replace y with u in (4-15) and (4-16).

In light of (4-14) define a coarser frequency grid:

$$\widetilde{\omega}_l = \frac{2\pi l(2m_T+1)}{T} + \frac{2\pi(m_TT+1)}{T} \qquad (4-19)$$

with $l = 0, 1, \cdots, L_T - 1$ where $L_T = \lfloor\frac{T}{2m_T+1}\rfloor$. Lemma 1 implies mean-square consistency of $\widehat{S}_{yu}(\alpha;k_l(T))$, hence it implies that (i.p. stands for in probability)

$$\lim{}_{T\to\infty}\widehat{S}_{yu}(\alpha;k_l(T)) = S_{yu}(\alpha;2\pi f_l) \text{ i.p.} \qquad (4-20)$$

$$\lim{}_{T\to\infty}\widehat{S}_{uu}(\alpha;k_l(T)) = S_{uu}(\alpha;2\pi f_l) \text{ i.p.} \qquad (4-21)$$

Convergence in (4-20)-(4-21) is uniform in f by virtue of Lemma 1. Using the estimated cyclic spectra we have an estimator of the system transfer function at frequency ω_k

$$\widehat{H}(e^{j\omega_k}) = \widehat{S}_{uu}^{-1}(k)\widehat{S}_{yu}(k) \qquad (4-22)$$

provided that $\widehat{S}_{uu}^{-1}(k)$ exists. If $S_{uu}^{-1}(\omega_k)$ exists (cf. (AS4) and (4-5)), then it follows from [11, Thm. 1.7, p. 24] that

$$\lim{}_{T\to\infty}\widehat{H}(e^{j2\pi f}) = \lim{}_{T\to\infty}\widehat{S}_{uu}^{-1}(\alpha;k(T))\widehat{S}_{yu}(\alpha;k(T))$$
$$= \ H(e^{j2\pi f}) \ \text{ i.p.} \qquad (4-23)$$

where $\lim_{T\to\infty} k(T)/T \ = \ f$. Convergence in (4-23) is uniform in f.

Remark 1. In the rest of the paper we use ω_k to denote a frequency on the coarse grid (4-19) and use λ_k to denote a fixed frequency independent of record length T. □

4.2. An Equation Error Formulation

It follows from the definition of $G(e^{j\omega})$ (cf. (1-4)) that

$$-\sum_{i=1}^{n_a} G(e^{j\lambda_k})a_ie^{-j\lambda_k i} + \sum_{i=1}^{n_b} b_ie^{-j\lambda_k i} \ = \ G(e^{j\lambda_k}) \qquad (4-24)$$

for any λ_k. Noting that a_i's and b_i's are real and $G(e^{j\omega})$ is, in general, complex-valued, we rewrite (4-24) after replacing $G(e^{j\lambda_k})$ with the true transfer function estimate $\widehat{H}(e^{j\lambda_k})$, as

$$-\sum_{i=1}^{n_a}\text{Re}\{\widehat{H}(e^{j\lambda_k})e^{-j\lambda_k i}\}a_i + \sum_{i=1}^{n_b}\text{Re}\{e^{-j\lambda_k i}\}b_i$$
$$= \ \text{Re}\{\widehat{H}(e^{j\lambda_k})\}, \qquad (4-25)$$

$$-\sum_{i=1}^{n_a}\text{Im}\{\widehat{H}(e^{j\lambda_k})e^{-j\lambda_k i}\}a_i + \sum_{i=1}^{n_b}\text{Im}\{e^{-j\lambda_k i}\}b_i$$
$$= \ \text{Im}\{\widehat{H}(e^{j\lambda_k})\}. \qquad (4-26)$$

Using distinct frequencies λ_k $(k = 1, 2, \cdots, L)$ over $(0, \pi)$ (one may choose this set from the coarse grid (4-19)), (4-25) and (4-26) may be rewritten in a matrix-equation form as

$$\mathbf{F}_T\theta \ = \ \mathbf{f}_T \qquad (4-27)$$

where θ is as in (3-3) and

$$\mathbf{f}_T = [\,\text{Re}\{\widehat{H}(e^{j\lambda_1})\}\ \vdots\ \text{Im}\{\widehat{H}(e^{j\lambda_1})\}\ \vdots\ \cdots$$
$$\vdots\ \text{Re}\{\widehat{H}(e^{j\lambda_L})\}\ \vdots\ \text{Im}\{\widehat{H}(e^{j\lambda_L})\}\,]^T, \qquad (4-28)$$

and $\mathbf{F}_T$ is a $(2L) \times (n_a+n_b)$ matrix composed of appropriate elements from the left-side of (4-25) and (4-26). An ordinary least-squares solution to (4-27) is given by

$$\widehat{\theta}_{1T} \ = \ (\mathbf{F}_T^T\mathbf{F}_T)^{-1}\mathbf{F}_T^T\mathbf{f}_T \qquad (4-29)$$

assuming that the inverse in (4-29) exists (see Remark 2 in Sec. 5 for conditions for its existence). A numerically well-conditioned solution is obtained via a singular value decomposition formulation.

The above least-squares formulation is equivalent to the following formulation. Choose θ to minimize the cost

$$J_T(\theta) \ := \ \frac{1}{L}\sum_{l=1}^{L}\left|A(e^{j\lambda_l};\theta)\widehat{H}(e^{j\lambda_l}) \ - \ B(e^{j\lambda_l};\theta)\right|^2 \qquad (4-30)$$

where

$$B(e^{j\lambda_l};\theta) \ = \ \sum_{i=1}^{n_b} b_ie^{-j\lambda_l}, \qquad (4-31)$$

$$A(e^{j\lambda_l};\theta) \;=\; 1 + \sum_{i=1}^{n_a} a_i e^{-j\lambda_l}. \qquad (4-32)$$

Define the estimator

$$\widehat{\theta}_{2T} \;=\; \arg\{\min_{\theta\in\Theta_C} J_T(\theta)\} \qquad (4-33)$$

where Θ_C is a compact set.

5. CONVERGENCE ANALYSIS

In this section we prove weak consistency of the proposed approach when the true system lies within the model set, i.e. when $H(q)$ is of the type $G(q)$. Let us denote $G(q)$ parametrized by θ as $G(q;\theta)$. Let θ_0, n_{a0} and n_{b0} denote the true values of θ, n_a and n_b, respectively, such that $H(q) = G(q;\theta_0)$. We first consider (4-29).

Theorem 1. Suppose that $H(q) = G(q;\theta_0)$ for some θ_0 with model orders n_{a0} and n_{b0}. Under the model assumptions (AS1)-(AS5), if $\min(n_a - n_{a0}, n_b - n_{b0}) = 0$ and $n_a + n_b \le L$, then the estimator (4-29) is weakly consistent (i.e. $\widehat{\theta}_{1T}$ converges i.p. to θ_0 as $T \to \infty$). •

Proof. Mimic the proof of Theorem 1 in [2]. □

Remark 2. It is shown in the proof of Theorem 1 of [2] that when $\min(n_a - n_{a0}, n_b - n_{b0}) = 0$, the inverse in (4-29) exists for large T. □

Next we consider (4-33) which is useful when $\min(n_a - n_{a0}, n_b - n_{b0}) \ge 0$.

Theorem 2. Suppose that $H(q) = G(q;\theta_0)$ for some θ_0 with model orders n_{a0} and n_{b0}. Under the model assumptions (AS1)-(AS5), if $\min(n_a - n_{a0}, n_b - n_{b0}) \ge 0$, $n_a + n_b \le L$, and $\theta_0 \in \Theta_C$, then the estimator $\widehat{\theta}_{2T}$ specified by (4-33) converges i.p. to a set $\mathcal{D}$ where

$$\mathcal{D} := \{\theta \mid B(q;\theta)/A(q;\theta) = H(q) = G(q;\theta_0)\}. \qquad (5-1)$$

Proof. Mimic the proof of Theorem 2 in [2]. Note that the convergence to the set $\mathcal{D}$ is to be interpreted in the sense of Ljung [5, p. 215]. □

When $\min(n_a - n_{a0}, n_b - n_{b0}) = 0$, $\mathcal{D} = \{\theta_0\}$.

6. SIMULATION EXAMPLE

This example is based upon [10]. The true open-loop plant $H(q)$ is given by

$$H(q) \;=\; \frac{q^{-1} + 0.5q^{-2}}{1 - 1.85q^{-1} + 0.525q^{-2}}\,; \quad \text{poles: } 1.5, 0.35\,.$$

The controller $F(q)$ is given by $F(q) = [0.35 - 0.28q^{-1}][1 - 0.8q^{-1}]^{-1}$. The closed-loop system is stable. We take

$$e(t) \;=\; \frac{1 - 1.795q^{-1} + 1.4328q^{-2} - 0.59608q^{-3} + 0.08738q^{-4}}{1 - 1.7q^{-1} + 0.33q^{-2} + 1.063q^{-3} - 0.6408q^{-4}}\,\epsilon(t)$$

where $\epsilon(t) \sim \mathcal{N}(0,\sigma^2)$ and i.i.d. Finally, the external input $v(t)$ is taken to be cyclostationary

$$v_f(t) \;:=\; \sqrt{2}\cos(3\pi t/8)\,\nu(t)$$

where $\nu(t) \sim \mathcal{N}(0,1)$, i.i.d. and independent of $\{\epsilon(t)\}$. This leads to $A_{\mathcal{M}} = \{0, 0.75\pi, 1.25\pi\}$. The cycle frequency at $\alpha = 0.75\pi$ which was selected for system identification via cyclic spectral analysis. The power σ^2 of $\{\epsilon(t)\}$ was scaled to achieve a closed-loop SNR of 14 (11.46 dB) where referring to (4-1)-(4-3) we define

$$\text{SNR} = \frac{\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E[s^2(t)]}{\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E[\eta^2(t)]}. \qquad (6-1)$$

The required cyclic spectra (auto and cross) were estimated via frequency-domain averaging of cyclic periodogram/ cross-periodogram using non-overlapping rectangular windows (see (4-14)). Table 1 shows the results of averages over 100 Monte Carlo runs based upon a record length $T = 2048$ with $2m_{2048} + 1 = 23$ in (4-14). Table 2 shows the same for a record length $T = 4096$ with $2m_{4096} + 1 = 45$.

TABLE 1 : results based on 100 Monte Carlo runs

Parameters	True	T = 2048 Mean	σ	RMS
a_1	−1.8500	−1.8501	0.0490	0.0490
a_2	0.5250	0.5491	0.0507	0.0562
b_1	1.0000	0.9793	0.0588	0.0624
b_2	0.5000	0.4638	0.0595	0.0698

TABLE 2 : results based on 100 Monte Carlo runs

Parameters	True	T = 4096 Mean	σ	RMS
a_1	−1.8500	−1.8448	0.0357	0.0361
a_2	0.5250	0.5297	0.0367	0.0370
b_1	1.0000	0.9862	0.0367	0.0392
b_2	0.5000	0.4902	0.0416	0.0427

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