ERRORS-IN-VARIABLES MODELLING IN OPTICAL FLOW PROBLEMS

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ABSTRACT

Although still in practice, the use of total least squares (TLS) in optical flow estimation is unreliable. TLS implicitly assumes that the error terms affecting the partial derivatives of the image intensities are independent. The usual methods for estimating the partial derivatives ensures that the errors are strongly correlated. Due to this correlation, an alternative method is required to treat the resulting errors-in-variables (EIV) problem. In this paper we propose a new method for estimating optical flow based on Sprent's procedure. This method incorporates a general EIV model and provides a far simpler computational procedure than found in previous solutions.

1. INTRODUCTION

Gradient-based optical flow estimation techniques rely on the assumption that the brightness or intensity of a particular point in a moving pattern does not change with time [4]. Using a Taylor's series expansion, this assumption leads to the brightness constraint equation:

$$g_1\nu_1 + g_2\nu_2 + g_3 = 0 \tag{1}$$

In (1) g_1, g_2 and g_3 are the partial derivatives of the image intensity with respect to spatial coordinates x_1, x_2 and time t. With ν_1, ν_2 being the unknown x_1, x_2 components of the optical flow. Since there are two unknowns in (1), further constraints are required to uniquely solve for the optical flow. One approach for constraining the solution is to fit measurements from a neighbourhood of surrounding pixels to a local model. This approach has been used in [5], [14], [15], [8], [3] and [12].

Let $\Omega = \{p_i : i = 1 \cdots m, m > 2\}$ denote a particular neighbourhood of m pixels which have been ordered lexicographically. Suppose that the optical flow is locally constant in Ω. If we can observe all partial derivatives without error, determining the optical flow only requires solving a linear system of equations. However due to noise introduced in the capturing process, the partial derivatives cannot realistically be observed without error and hence solving for the optical flow becomes a statistical estimation problem.

Using classical least squares (LS) to estimate the optical flow [5] is flawed. LS makes the implicit assumption that the two spatial derivatives, g_1 and g_2 are error free with all the uncertainties confined only to g_3. In statistics, situations for which g_1, g_2 are not error free are said to have "errors-in-variables" (EIV) and LS estimators for these cases are statistically inconsistent ([10],[6]).

In this paper we propose a new method for estimating optical flow which incorporates a general EIV model. Our method is based on the work of Sprent [10] and involves a much simpler computational procedure than previous attempts at addressing EIV.

In the next section we will set up an EIV model for our optical flow estimation problem, and in the following section present a review of previous work in this area. We then outline our method, implementation details and present simulation results in section 4, 5 and 6 respectively. Finally, a discussion of the results and future research plans is presented in section 7.

2. STOCHASTIC MODEL

Suppose we are interested in estimating optical flow from a gray-scale image sequence but the only measurements available are noisy image intensities. Let y_j be the observed intensity at pixel p_j such that:

$$y_j = g_o(p_j) + \epsilon_j \tag{2}$$

where g_o is the true image intensity and ϵ_j are iid Gaussian random variable with variance σ_ϵ^2.

In general the partial derivatives of the image intensity are obtained by multiplying an appropriate weight vector with $\mathbf{y}$, where $\mathbf{y}$ represents the lexicographically ordered noisy intensities over the support region Λ of size q pixels. That is, the partial derivative $g_i\left(p_j\right)$ is approximated using

$$\eta_i\left(p_j\right) = \mathbf{w}_{i,j}^T\,\mathbf{y} = \mathbf{w}_{i,j}^T\,\mathbf{g}_o + \mathbf{w}_{i,j}^T\,\boldsymbol{\epsilon} \tag{3}$$

where $\mathbf{w}_{i,j}^T$ is some appropriate vector of weights. We will further assume that $\mathbf{w}_{i,j}^T \mathbf{g}_o$ gives the true value of the partial derivative and hence we can re-write (3) as

$$\eta_i(p_j) = g_i(p_j) + \delta_{i,j} \tag{4}$$

where

$$\delta_{i,j} = \mathbf{w}_{i,j}^T \boldsymbol{\epsilon} \tag{5}$$

If we combine the weights vector together to form matrix $\mathbf{W}_i = [\mathbf{w}_{i,1}, \cdots, \mathbf{w}_{i,m}]$, our stochastic model in vector form is:

$$\begin{aligned} \mathbf{y} &= \mathbf{g}_o + \boldsymbol{\epsilon} \qquad \boldsymbol{\epsilon} \sim N_q\left(\mathbf{0}, \sigma_\epsilon^2 \mathbf{I}\right) && (6) \\ \boldsymbol{\eta}_i &= \mathbf{W}_i^T \mathbf{y} = \mathbf{g}_i + \boldsymbol{\delta}_i && (7) \\ \boldsymbol{\delta}_i &= \mathbf{W}_i^T \boldsymbol{\epsilon} && (8) \end{aligned}$$

3. PREVIOUS WORK

Synopses of current optical flow estimation techniques can be found in [2], [13] and [7]. In this section we will outline only those techniques which have attempted to address EIV.

3.1. Total Least Squares

The fact that partial derivatives of the image intensities cannot be observed without error has been recognised in [9], [14], [15] and more recently in [12]. All of these works applied total least squares (TLS) or TLS-based methods to treat EIV. TLS, however, implicitly assumes that all the errors affecting the partial derivatives $\delta_{i,j}$ are independent and identically distributed. TLS can be shown to be more accurate than LS under this assumption but TLS may be less accurate than LS if the assumption does not hold [1] .

As can be seen from equation (5), in optical flow problems $\delta_{i,j}$ are generally correlated as they are all derived from a common noise source $\boldsymbol{\epsilon}$. Thus in general, we do not expect TLS to give reliable estimates of the optical flow.

3.2. Nagel's Method

Nagel in [8] developed a maximum likelihood method which takes into account the correlation between the error terms $\delta_{i,j}$. The aim of Nagel's method is to maximise the joint probability of $\boldsymbol{\epsilon}$ subject to the constraint

$$\left(\nu_1 \mathbf{W}_1^T + \nu_2 \mathbf{W}_2^T + \mathbf{W}_3^T\right) \mathbf{g}_o = \mathbf{0}.$$

This is equivalent to minimising:

$$J = \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} + 2\boldsymbol{\lambda}\left(\nu_1 \mathbf{W}_1^T + \nu_2 \mathbf{W}_2^T + \mathbf{W}_3^T\right)(\mathbf{y} - \boldsymbol{\epsilon}) \tag{9}$$

by suitable choices of $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_m)^T$, ν_1 and ν_2. To solve this minimisation problem, Nagel describes an iterative method that successively improves estimates $\boldsymbol{\lambda}$, ν_1 , ν_2 and $\boldsymbol{\epsilon}$.

It should be noted that Nagel's method requires the estimation of $(m + q + 2)$ parameters. In most applications, estimates of $\boldsymbol{\lambda}$ and $\boldsymbol{\epsilon}$ do not serve any useful purpose and calculation can be quite time consuming if m or q is large.

4. PROPOSED METHOD

Our proposed method is based on the work of Sprent [10]. In the 1960's, Sprent developed an estimation procedure which can incorporate a general EIV model including the case of correlated error terms as encountered in optical flow estimation. To the author's knowledge, Sprent's procedure has not been previously applied to the estimation of optical flow.

In a statistical framework, estimating ν_1 and ν_2 can be viewed as estimating the parameters in the linear *functional* relationship between mathematical variables g_1, g_2 and g_3, given a set of measurements $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$ and $\boldsymbol{\eta}_3$ which satisfies equation (7). The functional relationship of (1) leads to a *structural* relationship between the observed variables such that:

$$\boldsymbol{\eta}_1 \nu_1 + \boldsymbol{\eta}_2 \nu_2 + \boldsymbol{\eta}_3 = \boldsymbol{\delta}_1 \nu_1 + \boldsymbol{\delta}_2 \nu_2 + \boldsymbol{\delta}_3 \tag{10}$$

Detailed discussions of functional and structural relationships are given in [6] and [11].

In [10], Sprent describes a generalised least squares approach to linear functional relationships. Sprent's procedure is to minimise over $\boldsymbol{\nu} = (\nu_1, \nu_2)$:

$$J = \mathbf{z}^T \boldsymbol{\Sigma}^{-1} \mathbf{z} \tag{11}$$

where

$$\mathbf{z} = \text{vector of residuals}$$

and

$$\boldsymbol{\Sigma} = var(\mathbf{z})$$

In optical flow estimation we let $\mathbf{z}$ be the residual of the brightness constraint,

$$\mathbf{z} = \boldsymbol{\eta}_1 \nu_1 + \boldsymbol{\eta}_2 \nu_2 + \boldsymbol{\eta}_3$$

Further let $\boldsymbol{\Theta}^T = \nu_1 \mathbf{W}_1^T + \nu_2 \mathbf{W}_2^T + \mathbf{W}_3^T$ then $\mathbf{z} = \boldsymbol{\Theta}^T \mathbf{y}$ and the covariance matrix of $\mathbf{z}$ is,

$$\begin{aligned} \boldsymbol{\Sigma} &= var(\mathbf{z}) = var\left(\boldsymbol{\Theta}^T \mathbf{y}\right) \\ &= var\left(\boldsymbol{\Theta}^T \mathbf{g}_o + \boldsymbol{\Theta}^T \boldsymbol{\epsilon}\right) = var\left(\boldsymbol{\Theta}^T \boldsymbol{\epsilon}\right) \\ &= \boldsymbol{\Theta}^T \boldsymbol{\Theta}\, \sigma_\epsilon^2 \end{aligned}$$

(Note that $\boldsymbol{\Theta}$ is a function of the required optical flow ν_1 and ν_2). Since $J \propto \sigma_\epsilon^2$, explicit knowledge of the variance is not required. Hence the minimisation problem can be written as:

$$J = \mathbf{z}^T \left(\boldsymbol{\Theta}^T \boldsymbol{\Theta}\right)^{-1} \mathbf{z} = \mathbf{y}^T \boldsymbol{\Theta} \left(\boldsymbol{\Theta}^T \boldsymbol{\Theta}\right)^{-1} \boldsymbol{\Theta}^T \mathbf{y} \tag{12}$$

which is a two dimensional non-linear optimisation problem.

It should be noted that Sprent in [10] does not prescribe a method for calculating the estimator when there is correlation between the measurements at different pixel sites. We have chosen to use a conjugate gradient based method which will be outlined in the next section.

5. IMPLEMENTATION

We have implemented a conjugate gradient method for solving the non-linear optimisation in equation (12). Let $\boldsymbol{\nu}_k$ be the solution at the kth iteration. An improved estimate is constructed by:

$$\boldsymbol{\nu}_{k+1} = \boldsymbol{\nu}_k + \alpha_k \mathbf{p}_k \tag{13}$$

where α_k is such that $J\left(\boldsymbol{\nu}\right)$ is minimum in the search direction $\mathbf{p}_k$. The next search direction is then given by:

$$\mathbf{p}_{k+1} = -\mathbf{d}_{k+1} + \beta_k \mathbf{p}_k \tag{14}$$

where $\mathbf{d}_{k+1}$ is the gradient of J at $\boldsymbol{\nu}_{k+1}$. In our implementation β_k is given by $\mathbf{d}_{k+1}^T \mathbf{d}_{k+1} / \mathbf{d}_k^T \mathbf{d}_k$. For our optical flow problem the gradient $\mathbf{d}$ is given by:

$$\mathbf{d} = \begin{bmatrix} 2\mathbf{s}^T \left(\boldsymbol{\eta}_1 - \boldsymbol{\Theta}^T \mathbf{W}_1 \mathbf{s}\right) \\ 2\mathbf{s}^T \left(\boldsymbol{\eta}_2 - \boldsymbol{\Theta}^T \mathbf{W}_2 \mathbf{s}\right) \end{bmatrix} \tag{15}$$

where

$$\mathbf{s} = \left(\boldsymbol{\Theta}^T \boldsymbol{\Theta}\right)^{-1} \mathbf{z} \tag{16}$$

Hence at each iteration the computation task is to solve a one-dimensional minimisation problem and to invert a $p \times p$ matrix in equation (16).

6. SIMULATION

In the simulation study, zero mean iid Gaussian noise was added to the translating tree sequence used in [2]. We have chosen to use a neighbourhood size of 20×20, and calculated partial derivatives using the simple differencing as used in [4]. For each SNR, optical flow estimates were calculated using LS, TLS and our method. Results for the 25 dB case can be seen in figures 1 and 2 and for the 20dB case in figures 3 and 4. The true optical flow can be seen in figure 5 for comparison.

In the 25dB case, there were neighbourhoods near the bottom left of the image for which the LS estimate detected little or no flow at all. Some regions of the TLS estimate gave large anomalous flows. These regions correspond to areas where the system of equations were highly inconsistent. In comparison, a fairly accurate estimate over the entire region was obtain using our method.

When the images were furthered degraded to a SNR of 20dB, the LS estimate detected almost no flow at all. More regions in the TLS estimate gave anomalous results, while the our method still performed adequately with errors predominantly occurring near the bottom left of the image. This region corresponds to the area under the branch where there is little intensity gradient information to aid the estimation of optical flow.

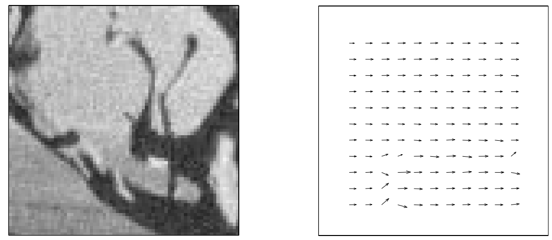


Figure 1: Subsection of the translating tree sequence used in simulation studies (left) and optical flow estimates using our method (right). (SNR 25dB).

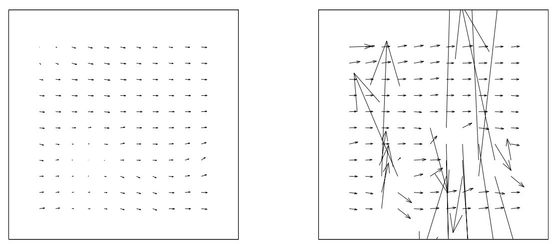


Figure 2: LS (left) and TLS (right) optical flow estimates (SNR 25dB).

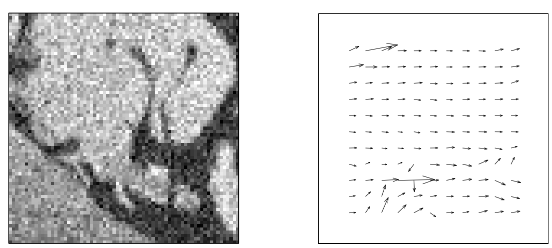
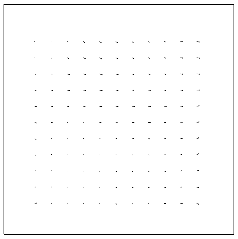


Figure 3: Subsection of the translating tree sequence used in simulation studies (left) and optical flow estimates using our method (right). (SNR 20dB).

7. DISCUSSION

Although still in current practice, the use of TLS in the estimation of optical flow is unreliable as TLS implicitly assumes that all the error terms affecting the partial derivatives



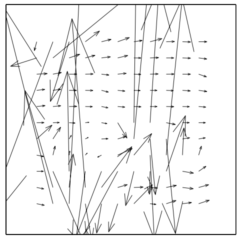


Figure 4: LS (left) and TLS (right) optical flow estimates (SNR 20dB).

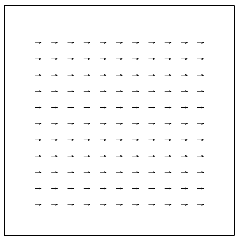


Figure 5: True optical flow for the translating tree sequence.

of the intensities are independent. As discussed in section 2, we cannot realistically expect that the errors to be uncorrelated since they are all derived from the same noise source ϵ. In our simulation study presented in section 6, we have shown that by incorporating a comprehensive EIV model, our new method for estimating optical flow is more accurate and can operate over a greater range of SNRs than either LS or TLS.

While Nagel's maximum likelihood method takes into account the correlation between errors in the measurements. Our technique based on Sprent's has the advantage that it requires neither estimation of the noise source ϵ or the Lagrange multipliers $\boldsymbol{\lambda}$ allowing an elegant implementation involving conjugate gradients.

It should be noted that there is a connection between Nagel's method and Sprent's procedure. The objective function in Nagel's method can be converted to the objective function in Sprent's by first concentrating out the incidental parameters. To the authors' knowledge neither Sprent's method nor a reformulation of Nagel's methods have previously been used in optical flow estimation. Sprent's procedure is not limited to the stochastic model we have used in this paper and in further work we will investigate the use of Sprent's for more complicated noise models.

8. REFERENCES

[1] T. J. Abatzoglou, J. M. Mendel, and G. A. Harada. The constrained total least squares technique and its applications to harmonic superresolution. *IEEE Transaction on Signal Processing*, 39(5):1070–1087, May 1991.

[2] J. L. Barron, D. J. Fleet, and S. S. Beauchemin. Systems and experiment: Peformance of optical flow techniques. *International Journal of Computer Vision*, 12(1):43–77, February 1994.

[3] A. Del Bimbo, P. Nesi, and J. L. C. Sanz. Optical flow computation using extended constraints. *IEEE Transcations on Image Processing*, 5(5):720–739, May 1996.

[4] B. K. P. Horn and B. G. Schunck. Determining optical flow. *Artificial Intelligence*, 17:185–203, 1981.

[5] J. K. Kearney, W. B. Thompson, and D. L. Boley. Optical flow estimation: An error analysis of gradient-based methods with local optimization. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, PAMI-9(2):229–244, March 1987.

[6] M. G. Kendall. *The Advanced Theory of Statistics*. Griffin, London, 3rd edition, 1969.

[7] A. Mitiche and P. Bouthemy. Computation and analysis of image motion: A synopsis of current problems and methods. *International Journal of Computer Vision*, 19(1):29–55, 1986.

[8] H. Nagel. Optical flow estimation and the interaction between measurment errors at adjacent pixel positions. *International Journal of Computer Vision*, 15:271–288, 1995.

[9] M. Shizawa and K. Mase. Simultaneous multiple optical flow estimation. In *Proceedings of the 10th International Conference on Pattern Recognition*, pages 274–278, Alantic City, New York, 1990.

[10] P. Sprent. A generalized least-squares approach to linear functional relationships. *Journal of Royal Statistical Society Series B*, 28(2):278–297, 1966.

[11] P. Sprent. *Models in Regression and related topics*. Methuen's monographs on applied probability and statistics. Methuen, London, 1969.

[12] S. Srinivasan and R. Chellappa. Robust modeling and estimation of optical flow with overlapped basis functions. Technical Report CAR-TR-845, Center for Automation Research, University of Maryland, College Park, MD 20742, December 1996.

[13] M. A. Tekalp. *Digital Video Processing*. Prentice Hall, 1st edition, 1995.

[14] S. Wang, V. Markandey, and A. Reid. Total least squares fitting spatiotemporal derivatives to smooth optical flow fields. In *Proceedings of the SPIE: Signal and Data Processing of Small Targets*, volume 1698, pages 42–55, 1992.

[15] J. Weber and J. Malik. Robust computation of optical flow in a multi-scale differential framework. *International Journal of Computer Vision*, 14:67–81, 1995.