FAST, ACCURATE SUBSPACE TRACKING USING OPERATOR RESTRICTION ANALYSIS

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ABSTRACT

A new noniterative subspace tracking method is presented. This method is called the operator restriction algorithm (OPERA) and it can be used whenever an update to the principal components of an EVD or SVD of a rank-one update of a given matrix is needed. The updating algorithms are based on the technique of restricting a linear operator to a subspace and the concept of an invariant subspace and its generalization, a pair of singular subspaces. The accuracy of the algorithm is comparable to an EVD or SVD. An application is made to bearing estimation of highly nonstationary sources. Flop counts, tracking accuracy and subspace accuracy for OPERA are compared with other fast algorithms and with the EVD.

1. INTRODUCTION

The bearing (DOA) estimation problem treated in this paper arises from J narrowband noncoherent plane waves impinging on a linear array of L sensors spaced one-half wavelength apart. The sensor outputs are subject to additive white Gaussian noise. The equation describing this scenario is

$$x(k) = Ds(k) + n(k), \quad k = 1, 2, \cdots \tag{1}$$

where $x(k)$, called the 'data snapshot', is an L-vector of array outputs at sampling time, k. The columns of $D = [d_1 \; d_2 \; \cdots \; d_J]$ are the steering vectors. Here,

$$d_i \stackrel{\text{def}}{=} [1 \;\; \exp(j\pi \sin(\theta_i)) \;\; \cdots \;\; \exp(j(L-1)\pi \sin(\theta_i))]^T \tag{2}$$

A basis for the signal subspace must then be found in order to use one of the parameter extraction techniques, such as MUSIC or ESPRIT. The most general (SVD) formulation of the OPERA algorithm is suitable for tracking principal left and right singular vectors and their associated singular values. The EVD version of the algorithm is suitable for tracking eigencomponents and will be discussed first.

2. INVARIANT SUBSPACE UPDATING

An invariant subspace, X, of a linear Hermitian operator, T, is defined to be one for which $TX \subset X$. (Any representation of T in a particular coordinate system will be denoted by a different symbol than T.) It can be shown that any invariant subspace is equivalent to an eigenspace, i.e. to a subspace spanned by a subset of the set of eigenvectors of the matrix. A reduction in the size of the matrix representation of T is possible by restricting T to an invariant subspace. To illustrate this process, choose the eigenvectors $v_i, \; i = 1, \cdots, P$ as the basis for the invariant subspace, $\tilde{X}$ of T, where $\tilde{X}$ is P-dimensional. Since $Tv_i = \lambda_i v_i, \; i = 1, \cdots, P$, in the coordinate system with the eigenvectors above as a basis for the invariant subspace $\tilde{X}$, the matrix representation of the linear operator, T, can be represented by a $P \times P$ matrix, $\tilde{A}$. Furthermore, in this coordinate system $\tilde{A}$ is a diagonal matrix: $\tilde{A} = \mathrm{diag}(\lambda_1, \cdots, \lambda_P)$. Now, consider a rank-one update to T: $T + xx^H$. Then, x can be written as: $x = Qx + (I-Q)x$, where Q is the orthogonal projection onto the invariant subspace $\tilde{X}$ and $I - Q$ is its orthogonal complement. Let $Qx = \sum_{i=1}^{i=P} \alpha_i v_i$. Define $z = (I-Q)x = \alpha_{P+1}\tilde{w}_{P+1}$ with $\alpha_{P+1} = \|z\|$ and $\tilde{w}_{P+1} = z/\|z\|$. Here, $\tilde{w}_{P+1}$ is in general not an eigenvector. Now, $\tilde{w}_{P+1}$ is assumed to lie in the kernel of T so that in the extended basis $\{v_1, \cdots, v_P, \tilde{w}_{P+1}\}$ for a new invariant subspace X ($\tilde{X} \subset X$), T has the representation: $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_P, 0)$. In addition, $x = [\alpha_1 \; \cdots \; \alpha_P \; \alpha_{P+1}]^T$ in this basis. So, the update $T + xx^H$ can be written in the alternate coordinate system as:

$$\begin{pmatrix} \lambda_1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda_P & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \\ \vdots \\ \\ \alpha_{P+1} \end{pmatrix} \begin{pmatrix} \bar{\alpha}_1 & \cdots & \bar{\alpha}_{P+1} \end{pmatrix} \tag{3}$$

If the application here is to subspace tracking for bearing estimation, x is the data snapshot. In general this data snapshot might contain a component lying in the complement of the initial invariant subspace $\tilde{X}$, and thus the increase in the size of $\tilde{A}$ by one to a $P+1 \times P+1$ matrix as in (3) is necessary in general, because of one or more of the following:

1. The snapshot is composed of element data which is corrupted with additive noise.
2. One or more sources are nonstationary and so the signal subspace is changing.
3. An additional source has appeared.

In the first and second cases, the rank of the covariance matrix doesn't change following an update; in the third case, the rank increases by one. For details of the algorithm, see the next section, which discusses the more general SVD version of OPERA.

3. SINGULAR SUBSPACE UPDATING

The notion of an invariant subspace can be generalized so as to treat the case of nonsquare matrices. Consider a matrix, $A \in C^{M \times N}$ and let $R = \min\{M, N\}$. Let the SVD of A be: $A = U\Sigma V^H$ where $U = [u_1, \cdots, u_M]$, $V = [v_1, \cdots, v_N]$ and $\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_R)$. The dimensions of Σ are $M \times N$. Let U_s and V_s denote the matrices consisting of the first P ($P \leq M, N$) columns of U and V, respectively. It is clear that

$$\mathcal{R}(AV_s) \subset U_s \quad \text{and} \quad \mathcal{R}(A^H U_s) \subset V_s \tag{4}$$

where $\mathcal{R}(A)$ denotes the column space of A. Thus, although U_s and V_s are not invariant subspaces of A, the inclusion relations above suggest that a generalization of the invariant subspace idea [1] may prove useful. We define:

Definition 1 *Let $A \in C^{M \times N}$ and let $X \in C^N$ and $Y \in C^M$ be subspaces of dimension P. Then, X and Y form a pair of singular subspaces for A if:*

1. *$AX \in Y$*
2. *$A^H Y \in X$*

Let the dimension of the principal subspace (i.e. signal subspace) be P. Let the left and right principal singular vectors be $u_1, \cdots, u_P$ and $v_1, \cdots, v_P$, respectively, and let the corresponding singular values be: $\sigma_1, \cdots, \sigma_P$. The matrix update is: $B \Rightarrow B + xy^H$. The goal is to calculate the principal singular vectors and singular values of the update in a coordinate system of reduced dimension. We have that $Bv_i = \sigma_i u_i,\ i = 1, \cdots, P$ and $B^H u_i = \sigma_i v_i,\ i = 1, \cdots, P$. Let $S_u = [u_1 \ \cdots, u_P]$ and let $S_v = [v_1 \ \cdots \ v_P]$. The vector x can be decomposed in the same manner as in the previous section as: $x = \sum_{i=1}^{P+1} \alpha_i u_i$ where $\alpha_{P+1} u_{P+1}$ ($\|u_{P+1}\| = 1$) is the component of x orthogonal to the left principal subspace. Let $U_x = [S_u \ u_{P+1}]$. In a similar manner, let y be written as: $y = \sum_{i=1}^{P+1} \beta_i v_i$ with $V_y = [S_v \ v_{P+1}]$ ($\|v_{P+1}\| = 1$).

In terms of the columns of U_x and V_y, the update of the data matrix can be written in this new coordinate system as:

$$\begin{pmatrix} \sigma_1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \sigma_p & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \\ \vdots \\ \\ \alpha_{P+1} \end{pmatrix} (\bar{\beta}_1 \quad \cdots \quad \bar{\beta}_{P+1}) \tag{5}$$

The difference between this equation and equation (3) is that the rank-one update here isn't necessarily Hermitian. As a consequence, there are two subspaces to be updated; a left principal subspace and a right principal subspace, as well as the associated singular values. It should be noted that of all the subspace updating algorithms presented here, excluding the full SVD, only the SVD version of OPERA is capable of tracking distinct left and right principal subspaces simultaneously.

There are 4 main tasks to be performed in one iteration of the SVD version of the OPERA algorithm; they are:

1. Calculating x and y in the new coordinate system.
2. Calculating the SVD of the update in the new coordinate system.
3. Calculating the updated principal singular vectors in the original coordinate system.
4. Retaining only those columns of the principal singular vectors which form a basis for the updated signal subspace.

These steps are as follows:

1. Let $w_0 = U_s^H x = [\alpha_1 \cdots \alpha_P]^T$, $w_1 = U_s w_0$ and $w_2 = x - w_1$ with $\alpha_{P+1} = \|w_2\|$. Then $\tilde{x}$ is x in the new coordinate system, with $\tilde{x} = [\alpha_1 \cdots \alpha_P \ \alpha_{P+1}]^T$. Similarly, let $v_0 = V_s^H y = [\beta_1 \cdots \beta_P]^T$, $v_1 = V_s y_0$ and $v_2 = y - v_1$ with $\beta_{P+1} = \|v_2\|$. Then $\tilde{y}$ is y in the new coordinate system, with $\tilde{y} = [\beta_1 \cdots \beta_P \ \beta_{P+1}]^T$.
2. The update equation in the transformed coordinate system is given above in equation 5. The task here is simply to calculate the SVD of the updated matrix in the transformed coordinate system.
3. Let the SVD of the updated matrix in the transformed coordinate system be:

   $$B + \tilde{x}\tilde{y}^H = U_1,\ _1 V_1^H \tag{6}$$

   The transformation back to the original coordinate system is simply:

   $$\tilde{U} = U_x U_1 \text{ and } \tilde{V} = V_y V_1 \tag{7}$$

   Here, $\tilde{U}$ is $M \times P$ and $\tilde{V}$ is $N \times P$. The updated singular values remain unchanged upon transformation.
4. One way to determine the dimension of the new subspace using a modelling aproach can be found in [2]. If the dimension of the subspace has been determined to be unchanged, the appropriate action is to drop the one column of $\tilde{U}$ and of $\tilde{V}$ corresponding to the smallest updated singular value.

Let $K = \max\{M, N\}$. Step 1 takes $O(KP)$ flops as shown above. In step 2 the SVD of a $P+1 \times P+1$ matrix is calculated, which takes $O(P^3)$ flops. The third step as shown above is $O(P^2 K)$. The overall complexity is then $O(P^2 K)$.

4. COMPARISON WITH OTHER ALGORITHMS

Although OPERA has similarities with the ROSE/ROSA class of methods [3], and with FAST [4], there are also significant differences, both conceptually and computationally.

Since OPERA updates the signal eigenspace at every iteration, the closest method in the ROSE/ROSENA class of methods is ROSE, where the signal eigenspace is tracked. The method of update for ROSE is an iterative one, however, [3] and very sensitive to the size of the perturbation when the signal subspace is nonstationary. The result is that the signal bearings are required to be slowly varying. In OPERA, this assumption is not needed because

OPERA is not a perturbation method and so any size update is acceptable. See Figures 2 and 3. It is important to note that OPERA does not average the signal sigenvalues. In addition, there is an SVD version of OPERA, whereas ROSE/ROSENA is restricted only to Hermitian matrices. In ROSE/ROSA a pairwise Gram-Schmidt orthogonalization is recommended to avoid stability problems, whenever large numbers of updates are needed. With OPERA, this is not necessary, since the signal subspace basis is derived directly from an SVD or EVD and is orthonormal to working precision; see Figures 4 and 5.

In FAST, the basis for the signal subspace being tracked is only approximately orthonormal, [4]. The principal singular values are not tracked, but reestimated at each step. The result is that for nonstationary scenarios, bearings and signal subspace bases as determined by the FAST algorithm can be inaccurate, as can be seen in Figures 3 and 5.

Another difference is that with FAST the data windowing as introduced in [4] is a sliding window and for an array with L sensors two of the steps in the algorithm vary linearly with w, the number of columns in the sliding window. As SNR decreases and the window size increases to compensate, the complexity of the algorithm becomes dominated by two terms varying linearly with w. As can be seen in Figure 1, the result is a larger flop count for smaller array sizes, compared to most of the other fast algorithms. In the SVD version of OPERA, with a sliding window, there are no steps that depend on the window size, so this is not a problem.

5. SIMULATIONS

The bearing estimation problem was presented in the introduction; here there are two sources and an 8-element array, so that $J = 2$ and $L = 8$. ESPRIT is used to find the bearings for all methods except ROSE, where Root-MUSIC is used (as is done in [3]). In Figure 1, a simulation showing the computational complexity for 8 subspace tracking methods is given. Given here are:

- EVD: full EVD.
- OPERA: EVD version of OPERA; this paper.
- FAST: FAST algorithm; see [4].
- PC: see [5].
- PROTEUS-2; see [6].
- PASTD; see [7].
- P-L (Prony-Lanczos); see [8].
- ROSE; see [9].

All the algorithms except FAST track the eigencomponents of the covariance matrix which is updated using an exponential window, i.e. $C \Rightarrow \alpha C + (1-\alpha)xx^H$. Here, SNR $=$ 0dB and $\alpha = 0.96$, which is a typical value when the SNR is low. The FAST algorithm uses a sliding window, with the window width equal to $[1/(1-\alpha)]$ where α is the forgetting factor for the exponential window and $[x]$ denotes the smallest integer larger than x.

It is important to note that many tracking algorithms perform poorly when the sources become sufficiently nonstationary. To illustrate this, use the previous array scenario. In order to remove any effects caused by additive

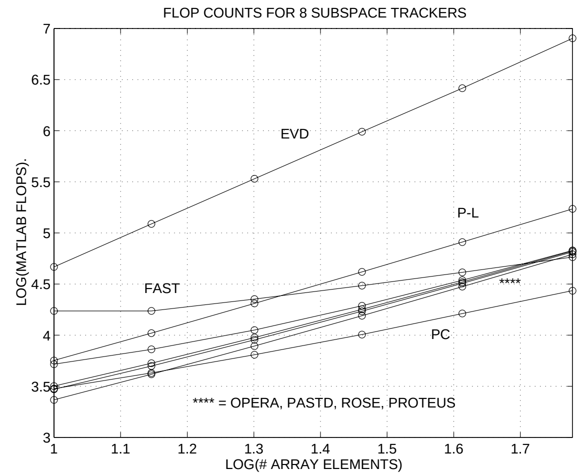

Figure 1: Flop Count vs Array Size for 8 Subspace Trackers.

noise, let the SNR be 10^{20} dB and assume exponential averaging with the forgetting factor $\alpha = 0.1$. For the FAST algorithm, let the number of columns in the sliding window be equal to 2. These windowings ensure that any resulting bearing errors are not due to a time lag caused by a large window size. Let there be two nonstationary signals, starting at 10 and 40 degrees respectively with each increasing at the rate of 0.4 degrees per iteration, so that the total angular movement over the 100 iteration trial is 40° for each of the sources. In Figures 2 and 3 it can be seen that the EVD, OPERA and P-L methods track the changing bearings most closely.

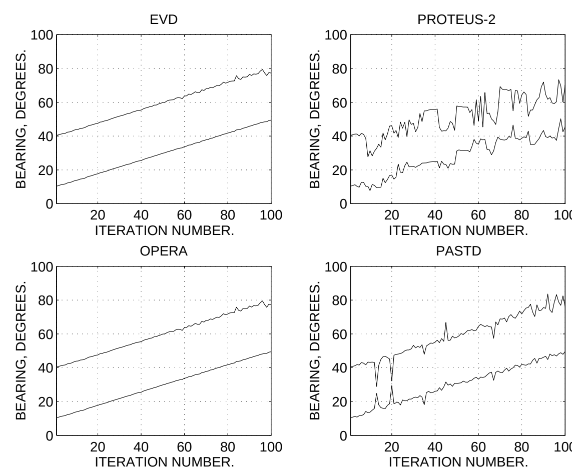

Figure 2: Bearings from EVD, PROTEUS-2, OPERA and PASTD.

In the third simulation the subspace errors for the same 8 methods are given in Figures 4 and 5. The subspace error is defined as the norm of the difference of the calculated and exact orthogonal projection matrices, onto the estimated and exact subspace, respectively. (Here, 'exact' is taken to mean 'no noise'.) In this example the SNR is 20dB, with

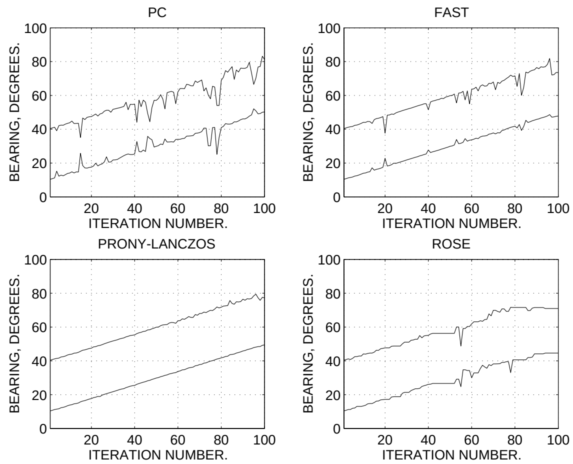

Figure 3: Bearings from PC, Prony-Lanczos, FAST and ROSE.

the two nonstationary sources starting and and increasing as before, with the same windowing specifications.

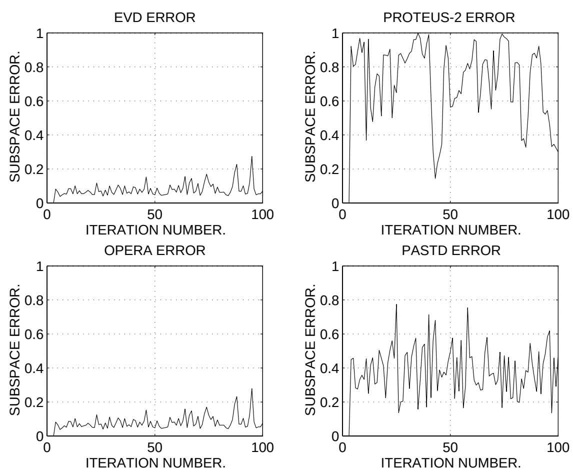

Figure 4: Subspace Errors: EVD, PROTEUS-2, OPERA and PASTD.

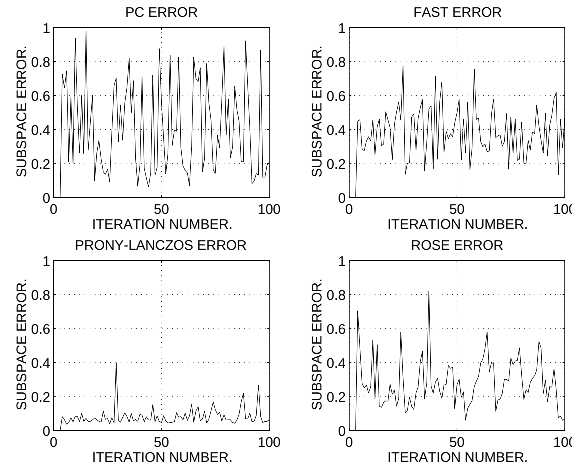

Figure 5: Subspace Errors: PC, Prony-Lanczos, FAST and ROSE.

6. CONCLUSION

The OPERA algorithm has been presented as a method for subspace tracking. The theoretical foundations for the EVD and SVD variations in the algorithm have been discussed, as well as conceptual and computational differences with other algorithms. From Figures 2-5 it can be seen that the EVD, OPERA and P-L algorithms are clearly superior in terms of bearing and subspace accuracy. Of these three algorithms, it can be seen from Figure 1 that OPERA is clearly the most efficient.

7. REFERENCES

[1] G. Stewart, "Error and Perturbation Bounds for Subspaces Associated with Certain Eigenvalue Problems," *SIAM Review*, vol. 15, pp. 727–764, October 1973.

[2] C. MacInnes and R. Vaccaro, "Tracking Directions-of-Arrivel with Invariant Subspace Updating," *Signal Processing*, vol. 49, 1996.

[3] R. DeGroat, "Noniterative Subspace Tracking," *IEEE Transactions on Signal Processing*, vol. 40, pp. 571–577, March 1992.

[4] D. Tufts, E. Real, and J. Cooley, "Fast Approximate Subspace Tracking (FAST)," in *Proceedings of the ICASSP 97 International Conference on Acoustics, Speech and Signal Processing*, (Munich, Germany), pp. 547–550, April 1997.

[5] B. Champagne, "Adaptive Eigendecomposition of Data Covariance Matrices Based on First-Order Perturbations," *IEEE Transactions on Signal Processing*, vol. 42, pp. 2758–2770, October 1994.

[6] B. Champagne and Q.-G. Liu, "Plane Rotation-Based EVD Updating Schemes for Efficient Subspace Tracking," *submitted to IEEE Transactions on Signal Processing*, August 1996.

[7] B. Yang, "Projection Approximation Subspace Tracking," *IEEE Transactions on Signal Processing*, vol. 43, pp. 95–107, January 1995.

[8] D. Tufts and C. Melissinos, "Simple, Effective Computation of Principal Eigenvectors and Their Eigenvalues and Application to High-Resolution Estimation of Frequencies," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 34, pp. 1046–1053, October 1986.

[9] R. DeGroat and R. Roberts, "Efficient, Numerically Stabilized Rank-One Eigenstructure Updating," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 38, pp. 301–316, February 1990.