DISTRIBUTIONS IN THE DISCRETE COHEN'S CLASSES

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ABSTRACT

Cohen's class of time-frequency distributions for continuous signals has recently been to extended to discrete signals using both an axiomatic approach and an operator theory approach. In this paper, we investigate the formulation of several classical time-frequency distributions (Wigner, Rihaczek, Margenau-Hill, Page, Levin, Born-Jordan, spectrogram) in the discrete Cohen's classes. The main result of this paper concludes that there does not exist a formulation of the Wigner distribution in all of the discrete Cohen's classes.

1. INTRODUCTION

There are four types of signals often used in signal processing, and to analyze these signals, there are four types of Fourier transforms. In Table 1 we list the four types of signals along with their properties and the appropriate Fourier transform. Since the Fourier transform is linear, under certain sampling conditions, the discrete Fourier transforms are samples of the continuous Fourier transform.

Cohen's class of time-frequency distributions [1, 2, 3] was originally formulated for type I signals. Recently, this class has been extended to the three types of discrete signals in Table 1 using both an axiomatic approach [4, 5, 6] and an operator theory approach [7, 8]. In addition, Richman et. al. [9] have used a group theory approach to derive type IV Wigner distributions.

However, since Cohen's class of time-frequency distributions are quadratic rather than linear, the discrete Cohen's classes are not generally sampled versions of the continuous Cohen's class. In this paper we will present results concerning discrete versions of several classical time-frequency distributions [1, 2, 3]: Wigner, Rihaczek, Margenau-Hill, Page, Levin, Born-Jordan, and the spectrogram. To simplify notation, we will use $x()$ to denote signals of all types and let the context indicate the type.

2. TYPE I COHEN'S CLASS

We will present Cohen's class in two different formulations:

$$\begin{aligned} &C^I(t,\omega;\psi) \\ &= \iint x(t_1)\, x^*(t_2)\, \psi(t_1-t, t_2-t)\, e^{-j\omega(t_1-t_2)}\, dt_1\, dt_2 \\ &\quad = \iint x(t'+\tfrac{\tau}{2})\, x^*(t'-\tfrac{\tau}{2})\, \phi(t-t',\tau)\, e^{-j\omega\tau}\, dt'\, d\tau \end{aligned}$$

The first provides simpler notation in the discrete case, but the second is more commonly used. The two forms of the kernel function, $\psi()$ and $\phi()$, are equivalent up to a 45 degree rotation.

We will use four properties, that are well defined for all four signal types, to provide an alternative definition of the classical Wigner distribution. One of these properties, called the time-reversal property, is not commonly used.

Definition. *Let $y(t) = x^*(-t)$ (and thus $Y(\omega) = X^*(\omega)$). A time-frequency distribution, T, is said to satisfy the time-reversal property if $T_y(t,\omega) = T_x(-t,\omega)$.*

Our alternative definition is based on the following theorem.

Theorem. *The type I Wigner distribution is the only time-frequency distribution for type I signals that: (i) is a quadratic function of the signal, (ii) is covariant to time shifts and frequency shifts, (iii) satisfies Moyal's formula, and (iv) satisfies the time-reversal property.*

Proof. Properties (i) and (ii) immediately limit us to TFDs in Cohen's class [3]. We will use both forms of Cohen's class in the proof, since the first is easier in the sequel and the first is more intuitive to most people. Property (iii) constrains the kernel to be of the form

$$\begin{aligned} \psi(t_1,t_2) &= \delta\big(\tfrac{-t_1-t_2}{2} - f(t_1-t_2)\big) \\ \phi(t,\tau) &= \delta\big(t - f(\tau)\big) \end{aligned}$$

for some function $f()$. Property (iv) constrains the kernel to be of the form

$$\begin{aligned} \psi(t_1,t_2) &= \psi(t_2,-t_1) \\ \phi(t,\tau) &= \phi(-t,\tau) \end{aligned}$$

Properties (iii) and (iv) are satisfied simultaneously if and only if $f(a) = 0$ for all a. This constrains the kernel to be

$$\begin{aligned} \psi(t_1,t_2) &= \delta\big(\tfrac{-t_1-t_2}{2}\big) \\ \phi(t,\tau) &= \delta\big(t\big) \end{aligned}$$

which corresponds to the type I Wigner distribution. $\square$

3. TYPE II COHEN'S CLASS

We will also present the type II Cohen's class in two different formulations:

$$
\begin{aligned}
&C^{II}(n,\omega;\psi)\\
&= \sum_{n_1}\sum_{n_2} x(n_1)\, x^*(n_2)\, \psi(n_1-n, n_2-n)\, e^{-j\omega(n_1-n_2)}\\
&= \sum_{n'\pm\frac{m}{2}, m\in \mathbf{Z}}\sum x(n'+\tfrac{m}{2})\, x^*(n'-\tfrac{m}{2})\, \phi(n-n', m)\, e^{-j\omega m}
\end{aligned}
$$

There is no general method for discretizing a distribution from the original Cohen's class to the discrete Cohen's class. However, we will use the obvious discretization when it seems appropriate. For example, the spectrogram and its obvious discretization:

$$
\begin{aligned}
S^I(t,\omega;h) &= \left| \int x(s)\, h(t-s)\, e^{-j\omega s}\, ds \right|^2\\
S^{II}(n,\omega;h) &= \left| \sum_m x(m)\, h(n-m)\, e^{-j\omega m} \right|^2
\end{aligned}
$$

are members of the type I and II Cohen's classes, respectively.

The kernels of the Rihaczek, Margenau-Hill, Page, Levin, and Born-Jordan distributions all have a straightforward discretization to the type II Cohen's class:

$$
\begin{aligned}
\phi_R(n,m) &= \delta\big(n-\tfrac{m}{2}\big)\\
\phi_{MH}(n,m) &= \delta\big(n-\tfrac{m}{2}\big)\,\delta\big(n+\tfrac{m}{2}\big)\\
\phi_P(n,m) &= \delta\big(n-\tfrac{|m|}{2}\big)\\
\phi_L(n,m) &= \delta\big(n+\tfrac{|m|}{2}\big)\\
\phi_{BJ}(n,m) &= \begin{cases} \frac{1}{|m+1|} & \text{for } |n| \le \frac{|m|}{2}\\ 0 & \text{otherwise}\end{cases}
\end{aligned}
$$

These kernels lead to a meaningful distribution in a broader sense. For example, compare the continuous Rihaczek distribution to the type II version defined using the above kernel:

$$
\begin{aligned}
R_x^I(t,\omega) &= x(t)\, X^*(\omega)\, e^{-j\omega t}\\
R_x^{II}(n,\omega) &= x(n)\, X^*(\omega)\, e^{-j\omega n}
\end{aligned}
$$

Note that the type II Rihaczek, Page and Levin distributions satisfy a type II version of Moyal's formula, and that the type II spectrogram, Born-Jordan, and Margenau-Hill distributions satisfy the time-reversal property.

If one applies the straightforward discretization of the type I Wigner kernel, then one obtains:

$$\phi(n,m) = \delta\big(n\big)$$

which results in the discrete Wigner distribution originally posed in [10]. If one compares the above definition to kernel sampling in Figure 1, then it is straightforward to see that the kernel has non-zero values only for odd values of m (lags). This is clearly undesirable, and does not present a useful definition of a type II Wigner distribution [1].

[1] If one oversamples the signal by a factor of two, then it is possible to obtain samples of the type I Wigner distribution, but this representation is not a member of the type II Cohen's class.

Since the discretization method does not provide a satisfactory result, we will use the above theorem in an attempt to define a type II Wigner distribution. Properties (i) and (ii) restrict us to TFDs in the type II Cohen's class. A TFD in the type II Cohen's class will satisfy Moyal's formula if and only if

$$\phi(n,m) = \delta\big(n - f(m)\big)$$

and will satisfy the time reversal property if and only if

$$\phi(n,m) = \phi(-n,m)$$

However, it is impossible to satisfy both of these kernel constraints simultaneously. To see this, refer to Figure 1a and note that for odd m there is no sample at $n=0$.

There exist many distributions in the type II Cohen's class that satisfy Moyal's formula (e.g. Rihaczek, Page, Levin), and there exist many distributions in the type II Cohen's Class that satisfy the time-reversal property (e.g. Margenau-Hill, Born-Jordan, spectrogram), but there do not exist *ANY* that satisfy both properties. Since type III signals are the dual of type II signals, the type III Wigner distribution also does not exist under this definition.

4. TYPE IV COHEN'S CLASS

For ease of notation, we will present the type IV Cohen's class in only one form:

$$
\begin{aligned}
&C^{IV}(n,k;\psi)\\
&= \sum_{n_1}\sum_{n_2} x(n_1)\, x^*(n_2)\, \psi(n_1-n, n_2-n)\, e^{-j2\pi k(n_1-n_2)/N}
\end{aligned}
$$

where N is the period of the signal and the kernel. In Figure 1 we show the kernel sampling structure for odd and even values of N.

Of the classical time-frequency distributions mentioned above, only the spectrogram, Rihaczek, and Margenau-Hill distributions have obvious discretizations to the Type IV Cohen's class. For example, the type IV Rihaczek kernel and distribution are formulated as:

$$
\begin{aligned}
\psi_R(n_1,n_2) &= \delta\big(n_1\big)\\
R_x^{IV}(n,k) &= x(n)\, X^*(k)\, e^{-j2\pi kn/N}
\end{aligned}
$$

Because of the periodicity of the kernel it is not obvious how to implement the Born-Jordan, Page, and Levin distributions, but we will not investigate this issue further in this paper.

As for the type II Cohen's class, there is no clear method for discretizing the Wigner distribution to the type IV Cohen's class. Richman et al. have recently defined Wigner distributions for type IV signals using group theory [9]. Surprisingly, the mathematical structure of the problem forced them to use two different groups for defining their type IV Wigner distributions for even and odd length signals. In addition, the properties of the distributions for even and odd length signals are quite different. For example, in Figure 2 we show the distributions defined by Richman for two chirp signals of length $N=127$ and $N=128$. Because of these differences we will attempt to define type IV Wigner distributions for even and odd length signals using the above theorem.

For type IV signals with an odd length, there exists exactly one type IV TFD that satisfies the four properties in the theorem.

This type IV TFD is equivalent to the definition given by Richman et al., and we will denote it as the type IV Wigner distribution (for odd length signals). The kernel of this type IV Wigner distribution is:[2]

$$\psi(n_1, n_2) = \delta\big(-(n_1 + n_2)(2^{-1})\big)$$

Figure 1b shows the kernel corresponding to the Wigner distribution for a signal that has a period of 3 samples. In Figure 1b the open circles correspond to a kernel value of 1, and the filled circles correspond to a kernel value of 0. To see that this is the only kernel that satisfies the four properties, follow the steps outlined in the proof of the theorem and note that the kernel is now a periodic function.

For type IV signals with an even length, there do not exist any Type IV TFDs that satisfy the four properties in the above theorem. There exist many distributions in the type IV Cohen's class for even length signals that satisfy Moyal's formula (Rihaczek) and many that satisfy the time-reversal property (spectrogram, Margenau-Hill) but none that satisfy both. To see this, follow the steps in the proof of the theorem and examine the kernel for odd m. Surprisingly, the length of the period determines whether or not all four properties can be satisfied. The definition proposed by Richman et al. for even length signals [9] does satisfy Moyal's formula, but does not satisfy the time-reversal property.

5. COMMENTS ON ALIASING

The properties of cross terms in the discrete Cohen's classes are different from the properties of cross terms in the continuous Cohen's class [4]. Several authors have attributed the cause of this to aliasing [11, 12]. Aliasing implies a loss of information, but since there exist many distributions in the type II and IV Cohen's classes that satisfy Moyal's formula, we know that there is no information lost in these distributions [13]. Also, if the differences between the continuous and discrete Cohen's classes were due to aliasing, then these differences should disappear as the signal sampling rate increases. However, this is not the case.

6. CONCLUSIONS

In this paper we present the formulation of several classical time-frequency distributions in the discrete Cohen's classes. Since there is no clear method for discretizing the Wigner distribution, we propose an alternative definition for the Wigner distribution that generalizes easily to discrete signals. Under this definition, we show that the Wigner distribution only exists for type I signals and for type IV signals with an odd length period. The former case corresponds to the classical definition, and the latter case corresponds to the definition given by Richman et al. While other definitions of the Wigner distribution are certainly possible, this result suggests why previous methods have failed to produce a satisfactory result for type II signals and for type IV signals with an even length period.

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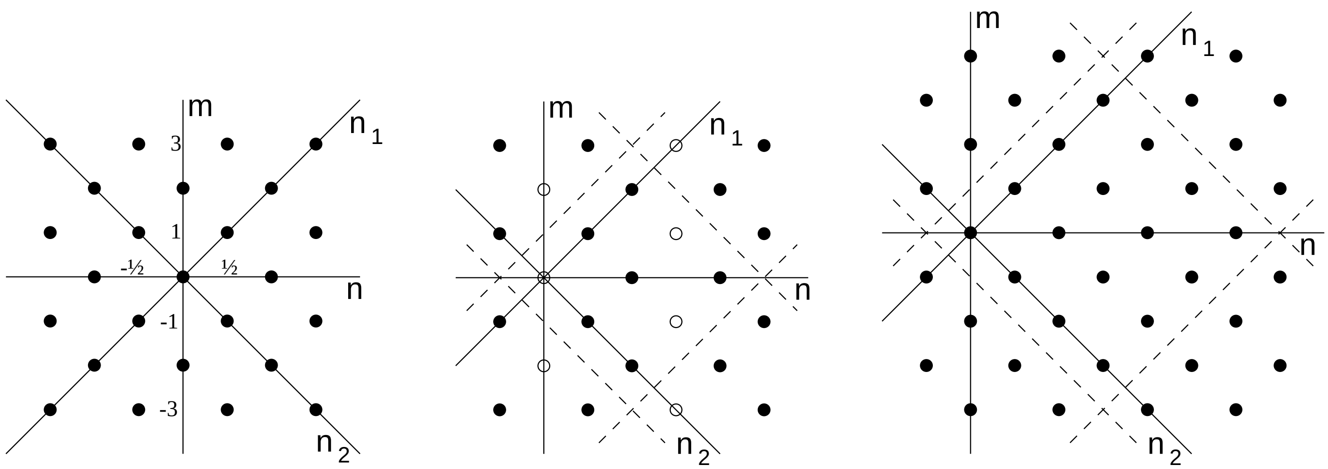
[2] Note that 2^{-1} is defined on the group of integers modulo N. For odd N, $(2^{-1}) = \frac{N+1}{2}$.

Table 1: Four types of signals and their properties in the time domain.

Type	Time Domain	Frequency Domain	Transform
I	continuous, aperiodic	continuous, aperiodic	Fourier transform
II	discrete, aperiodic	continuous, periodic	discrete-time Fourier transform
III	continuous, periodic	discrete, aperiodic	Fourier series
IV	discrete, periodic	discrete, periodic	discrete Fourier transform

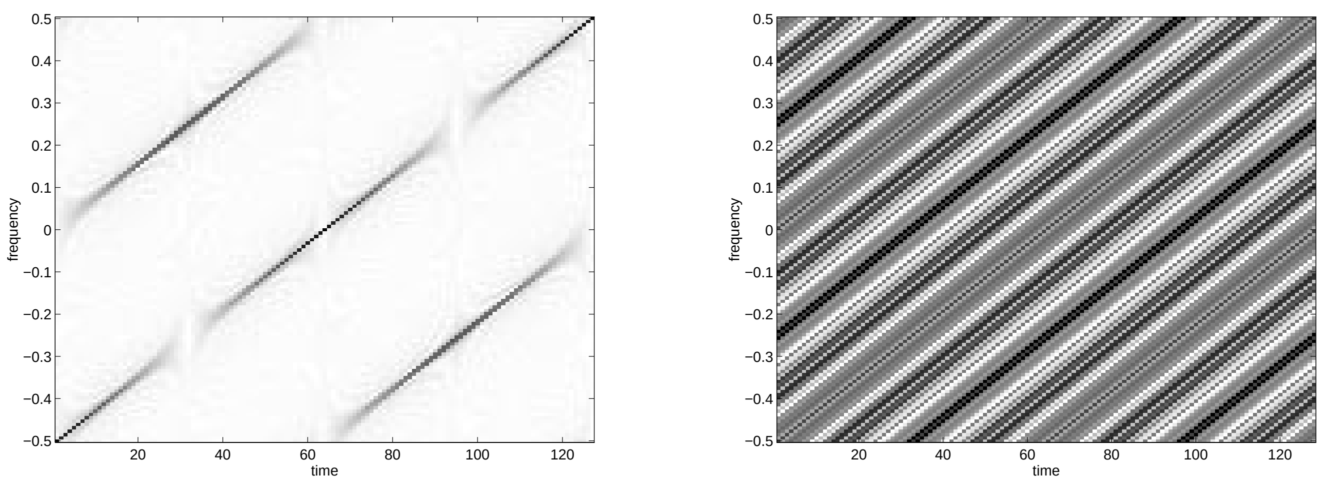


(a) type II

(b) type IV - odd length

(c) type IV - even length

Figure 1: Kernel sampling grids for type II and type IV signals. The solid lines denote the two axis systems and the dashed lines denote one period.



(a) type IV Wigner distribution - odd length linear chirp

(b) Richman distribution - even length linear chirp

Figure 2: Comparison of the two distributions defined by Richman for a chirp signal.