SUBSPACE DOMAIN FORWARDS-BACKWARDS AVERAGING

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ABSTRACT

In this paper a procedure which filters out roughly half of the array manifold errors for approximately centro-symmetric arrays is described. The procedure - subspace domain forwards -backwards (f/b) averaging - improves the performance of subspace based direction finding (DF) algorithms such as MUSIC and ESPRIT. Experimental data from the Mountaintop system are used to confirm the theoretical results.

1. INTRODUCTION

For centro-symmetric sensor arrays (arrays with 180^o rotational symmetry) there exists an invariance that allows each data vector to be used twice for covariance estimation in a process known as f/b averaging. Applications of f/b averaging include an improved linear prediction estimator [1], improving covariance estimation for adaptive beamforming and space-time adaptive processing [2-3], and in conjunction with spatial smoothing, a scheme to decorrelate coherent signals incident on an array for direction finding (DF) [4]. However, in practice the array may not be exactly centro-symmetric - there is some level of array manifold errors present. Previously, it was demonstrated that unknown array manifold errors have a minimal effect on the performance of adaptive beamformers utilizing f/b averaging [2].

Here the effect of f/b averaging in the presence of array manifold errors on superresolution DF algorithms such as MUSIC [5,6] is investigated. Conventional data domain f/b averaging in the presence of array manifold errors is analyzed and subspace domain f/b averaging is proposed as a way of reducing the effects of array manifold errors. Simulated results and experimental data which confirm the phenomena described are presented.

2. DATA MODEL

It is assumed the reader is familiar with the array signal processing signal model. This section is intended to introduce the notation used later. Consider a uniform linear array of N sensors with inter-element spacing d, though the results given here may be extended to any centro-symmetric array. The transfer function between direction of arrival (DOA) θ and the array's output is represented by the Vandermonde steering vector

$$\boldsymbol{a}(\theta) = [e^{j(\frac{1-N}{2})2\pi d\lambda^{-1}\sin(\theta)}, ..., e^{j(\frac{N-1}{2})2\pi d\lambda^{-1}\sin(\theta)}]^T \quad , \tag{1}$$

where λ represents wavelength. Note that the phase reference for $\boldsymbol{a}(\theta)$ is chosen to be at the physical center of the array and that

$$\boldsymbol{a}(\theta) = \boldsymbol{J}\boldsymbol{a}(\theta)^* \quad , \tag{2}$$

where $\boldsymbol{J}$ - the exchange matrix - has ones on the anti-diagonal and zeros elsewhere. The data received at the array output is the sum of the K incident signals and the noise

$$\boldsymbol{x}(t) = \sum_{k=1}^{K} \alpha_k(t)\boldsymbol{a}(\theta_k)+\boldsymbol{n}(t) \quad , \tag{3}$$

where $\alpha_k(t)$ is the complex amplitude of the kth signal and $\boldsymbol{n}(t)$ the noise vector at snapshot (i.e. time sample) t. We assume both the noise and $\alpha_k(t)$ to be a zero-mean, complex, Gaussian random processes, and the noise to be spatially white. Also, let μ_k be the signal to noise ratio (SNR) per array element of the kth signal.

The steering vector $\boldsymbol{a}(\theta)$ represents the desired array response for which the array was designed - a **presumed** array manifold - which due to modeling errors differs from the **true** manifold. The true manifold, $\boldsymbol{a_h}(\theta)$, may be related to $\boldsymbol{a}(\theta)$ by taking the Hadamard product of the latter with a vector of error coefficients, $\boldsymbol{h}=[h_1,...,h_N]^T$ which preserves the output power of the steering vector, i.e.

$$\boldsymbol{a}_h(\theta) = \boldsymbol{a}(\theta) \odot \boldsymbol{h} \text{ and } \boldsymbol{a_h}(\theta)^H\boldsymbol{a_h}(\theta) = \boldsymbol{a}(\theta)^H\boldsymbol{a}(\theta) \quad . \tag{4}$$

Except for the special case where $\boldsymbol{h} = \boldsymbol{J}\boldsymbol{h}^*$, $\boldsymbol{a_h}(\theta) \neq \boldsymbol{J}\boldsymbol{a_h}(\theta)^*$. This inequality (i.e. the true array is not exactly centro-symmetric) is assumed for the remainder of the paper. By inserting $\boldsymbol{a_h}(\theta)$ into Equation (3), data vectors for the actual array are obtained. The elements of $\boldsymbol{h}$ have the form

$$h_i = c + g_i \quad , \tag{5}$$

where c is a real constant and g_i is an error term with a zero-mean complex Gaussian distribution. Let the ratio of the variance of g_i to c be ξ^2. This model is used so that some expectations can be computed and used to give a 'feel' for how performance varies as a function of ξ. Generally the size of the array manifold errors - the difference between the true and presumed array manifolds - is small, i.e. $\xi^2 << 1$. We also assume

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that the array manifold errors are a random function of the DOA. The correlation of the array manifold errors over some arbitrarily small change in DOA is ignored here, but is open for future investigation.

The exact covariance matrix is defined as $\boldsymbol{R}=E\{\boldsymbol{x}(t)\boldsymbol{x}(t)^H\}$, where $E\{.\}$ represents the expectation operator. The sample covariance matrix is traditionally estimated from L snapshots (samples) of the data, i.e.

$$\hat{\boldsymbol{R}} = \frac{1}{L}\sum_{i=1}^{L}\boldsymbol{x}(t_i)\boldsymbol{x}(t_i)^H \quad . \tag{6}$$

For centro-symmetric arrays an improved estimate of the covariance matrix is obtained through f/b averaging, i.e.

$$\hat{\boldsymbol{R}}_f = \frac{1}{2L}\sum_{i=1}^{L}\boldsymbol{x}(t_i)\boldsymbol{x}(t_i)^H + \boldsymbol{x}^r(t_i)\boldsymbol{x}^r(t_i)^H = \tfrac{1}{2}\left(\hat{\boldsymbol{R}} + \boldsymbol{J}\hat{\boldsymbol{R}}^*\boldsymbol{J}\right) \tag{7}$$

where $\boldsymbol{x^r}(t)=\boldsymbol{Jx}(t)^*$. It is easily shown that the signals and the noise in the forwards ($\boldsymbol{x}(t)$) and backwards ($\boldsymbol{x^r}(t)$) samples are uncorrelated. The expected value of the forwards and backwards noise components from Equation (3) is $E\{\boldsymbol{Jn}(t)^*\boldsymbol{n}(t)^H\} = E\{\boldsymbol{Jn}(t)\boldsymbol{n}(t)^T\}^* = 0$ for any zero mean random process. Thus sample support is effectively doubled by the use of f/b averaging. Similarly if the modulating signal in Equation (3) is considered, the correlation between the forwards and backwards samples of the kth signal is $E\{\alpha_k(\alpha_k^*)^*\} = E\{\alpha_k\alpha_k\} = 0$ since α_k is also assumed to be zero mean.

3. EFFECT OF ARRAY MANIFOLD ERRORS

Suppose that due to the array manifold errors the array is not perfectly centro-symmetric, then the (apparently uncorrelated) forwards and backwards samples of a signal appear to have different steering vectors. Thus, a covariance matrix produced using f/b averaging will have two signal subspace eigenvectors / eigenvalues for each signal present. The properties of this enlarged signal subspace will be explored theoretically for the case of a single signal and a known covariance matrix.

Consider the signal covariance matrix of two uncorrelated signals with steering vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ and SNRs μ_1 and μ_2. The signal subspace eigenvectors ($\boldsymbol{e}_1$,$\boldsymbol{e}_2$) and eigenvalues (λ_1,λ_2) are given by [2,7];

$$\boldsymbol{e}_i \propto \beta\boldsymbol{a_1} + \gamma\boldsymbol{a_2}\,; \quad i = 1,2 \quad , \tag{8a}$$

$$\lambda_i = \tfrac{1}{2}N(\mu_1+\mu_2)\left[1 \pm \sqrt{1-\frac{4\mu_1\mu_2(1-|\psi|^2)}{(\mu_1+\mu_2)^2}}\,\right] \quad \text{and} \tag{8b}$$

$$\frac{\gamma}{\beta} = \frac{\lambda_i - \mu_1 N}{\mu_1 N\psi} \quad . \tag{8c}$$

where ψ is the cosine of the angle between $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. When the two signals are the forwards and backwards spatial samples, with steering vectors $\boldsymbol{a_h}$ and $\boldsymbol{a_h}^r$, each with SNR equal to μ, Equation (8b) simplifies to

$$\lambda_i = N\mu(1 \pm |\psi|) \quad . \tag{9}$$

The $\pm$ in Equation (9) corresponding to the first and second eigenvalues. For small levels of the array manifold errors, (i.e. $\xi^2 \ll 1$), then $|\psi|$ is close to 1, $\lambda_1 \approx 2N\mu$, and from Equation (8c)

$$\frac{\gamma}{\beta} \simeq \frac{1}{\psi} \approx 1 \quad . \tag{10}$$

Thus the first eigenvector of the covariance matrix is proportional to $\boldsymbol{a_h} + \boldsymbol{a_h}^r$, i.e. it depends on the average of the forwards and backwards steering vectors for a signal. Since the array manifold errors have been modeled as random from element to element, this averaging reduces the effect of the array manifold errors by 3 dB. For a single signal incident on a 10 element array, Figure 1 shows the 'distance' (1 minus the square of the cosine of the angle) between the presumed array manifold and the first eigenvector of the conventional and f/b averaged covariance matrices as a function of ξ^2. The result is averaged over 100 Monte Carlo runs - each with a different realization of the random array manifold errors. For small values of ξ^2 the primary eigenvector for the f/b averaged covariance matrix lies about 3 dB closer to the presumed array manifold as expected.

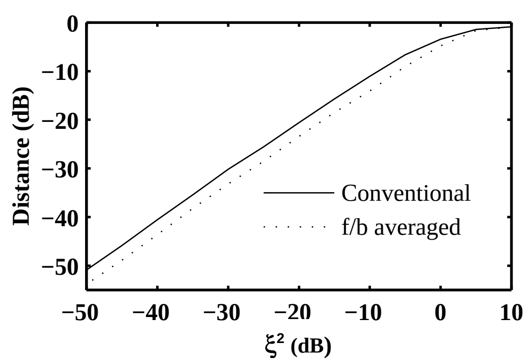

Figure 1: Plot of the 'distance' between the primary eigenvector and the presumed array manifold as a function of the size of the array manifold errors.

Now we consider the second eigenvalue/eigenvector pair of the f/b averaged covariance matrix. For the model of section 2 and small ξ^2, $E\{|\psi|\}$ is approximated as

$$E\{|\psi|\} = E\left\{\frac{|\boldsymbol{a}(\theta)_h^H\boldsymbol{a}(\theta)_h^{*r}|}{|\boldsymbol{a}(\theta)_h^H\boldsymbol{a}(\theta)_h|}\right\} \simeq \frac{1}{1+\xi^2} \approx 1-\xi^2 \quad . \tag{11}$$

This is also borne out by inspection of Figure 1. Thus the second eigenvalue of the covariance matrix is approximated as

$$\lambda_2 \approx N\mu\xi^2 \quad . \tag{12}$$

Substituting Equation (12) in to Equation (8c), for small ξ^2 the weighting of the steering vectors for the second eigenvector is

$$\frac{\gamma}{\beta} \simeq \frac{\xi^2 - 1}{\psi} \approx -1 \quad . \tag{13}$$

Thus the second steering vector of the covariance matrix depends on the difference (due to the array manifold errors) of the forwards and backwards spatial samples. Due to the random model of the array manifold errors used, it is clear the difference is also random in nature.

So, after f/b averaging with array manifold errors the signal subspace contains two sets of eigenvector / eigenvalue pairs, 'purified' pairs corresponding to the signals of interest (**purified subspace**), and random pairs due to the array manifold errors (**error subspace**). Ideally, only the purified subspace should be used for estimating the signal's DOA, since typical estimators are slightly degraded by the perturbation caused by including random 'noise like' eigenvectors in the signal subspace. However, methods such as AIC and MDL [8], (typically used to estimate the dimension of the signal subspace), cannot distinguish between the purified and error subspaces.

There are three scenarios which may result from f/b averaging with array manifold errors:

Scenario 1: Error eigenvalues are below the noise level. From Equation (12), if $\xi^{-2}<\mu N$, then the error eigenvalues will be below the noise floor. In this case the DOA estimator ought to give improved performance, since only the purified subspace is used.

Scenario 2: Error eigenvalues are above the noise level, but smaller than all the purified eigenvalues. If only the purified subspace is used in the DOA estimator (possible if say the number of signals is known a priori, or if AIC/MDL was used prior to f/b averaging) improved performance results. If both the purified and error subspaces are used in the DOA estimator, then performance is slightly degraded, since the signal subspace has now been perturbed by some random eigenvectors.

Scenario 3: Some error eigenvalues are larger than some of the purified eigenvalues. Since the purified and error subspaces are not easily separated, reduced performance results, either due to the DOA estimator incorrectly using the both the purified and error subspaces, or (if the number of signals is known) because not all of the purified subspace is used.

Figure 2 shows the results of a simulation corresponding to scenario 3. There are 2 signals incident on a 10 element array from $\sin(\theta)=\pm0.05$, with SNRs of 30 dB and 10 dB. The array manifold errors are set at $\xi^{-2}=-15$ dB, thus the error eigenvalue for the 30 dB SNR signal is at ~25 dB while the purified eigenvalue for the 10 dB signal is only at ~17.7 dB. 5 snapshots of data were available for covariance estimation in each case. Since the purified and error subspaces cannot be separated in any straightforward fashion, both, (i.e. 4 eigenvectors) are required in order for both signals to be resolved as is evident in Figure 2. Even so, performance is still worse with f/b averaging than without. The estimate variances for the 2 signals are 0.001 and 0.193 beamwidths with f/b averaging (4 signal subsapce eigenvettors)verses 0.0007 and 0.0430 beamwidths without.

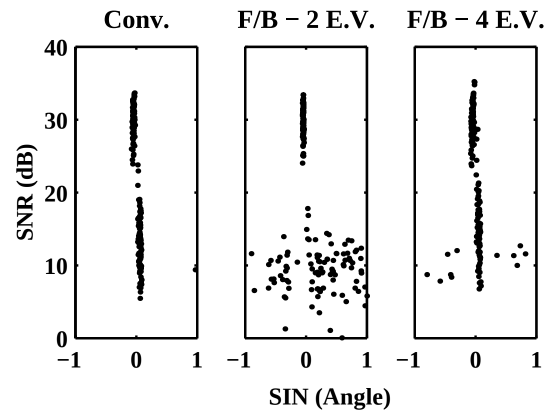

Figure 2: DF results for closely spaced signals with different SNRs. After f/b averaging, if only the first 2 signal subspace eigenvectors are used, then only one signal is resolved. In all cases there are some outling signal estimates

4. SUBSPACE DOMAIN F/B AVERAGING

If for scenario 3 above the effect of the signal power on the eigenvalues can be negated, then separating the purified and error subspaces would be easier. This may be accomplished by performing f/b averaging on the signal subspace, rather than on the data. The procedure we propose is as follows:

1) Estimate the covariance matrix, $\hat{\boldsymbol{R}}$, from the data
2) Find the signal subspace ($\boldsymbol{E}_s$) of $\hat{\boldsymbol{R}}$, (by eigen decomposition and use of AIC/MDL). Let the rank of $\boldsymbol{E}_s$ be r.
3) Form the projection matrix of the signal subspace $\boldsymbol{P}=\boldsymbol{E}_s\boldsymbol{E}_s^H$.
4) Apply f/b averaging $\boldsymbol{P}$, i.e. $\boldsymbol{P}_f = \frac{1}{2}(\boldsymbol{P}+\boldsymbol{J}\boldsymbol{P}^*\boldsymbol{J})$.
5) Eigen decompose $\boldsymbol{P}_f$ and take the r largest eigenvectors, (those corresponding to the r largest eigenvalues). This is the purified signal subspace $\boldsymbol{E}_p$. Use $\boldsymbol{E}_p$ in the direction finding algorithm.

It is easily proven that f/b averaging can be applied to $\boldsymbol{P}$ in the same way as it can be applied to $\hat{\boldsymbol{R}}$. Consider the eigen decomposition of the signal only covariance matrix $\boldsymbol{R}_s=\boldsymbol{A}\boldsymbol{S}\boldsymbol{A}^H=\boldsymbol{E}\Lambda\boldsymbol{E}^H$, where $\boldsymbol{A}$ is the matrix of steering vectors for the r signals present, $\boldsymbol{S}$ is the diagonal matrix of the signal powers of the (uncorrelated) signals, $\boldsymbol{E}$ is the matrix of the r eigenvectors of $\boldsymbol{R}_s$ and Λ the diagonal matrix of eigenvalues. Since $\boldsymbol{P}=\boldsymbol{E}\boldsymbol{E}^H$, the projection of the signal subspace is the same as the signal only covariance matrix with all of its signal subspace eigenvalues equal to unity. Since for any combination of signal DOAs, their signal powers may be adjusted so that the eigenvalues of $\boldsymbol{R}_s$ are unity, and f/b averaging applied, it follows the f/b averaging may also be applied to $\boldsymbol{P}$.

When f/b averaging is applied to $\boldsymbol{P}$ in the presence of array manifold errors, the rank of $\boldsymbol{P}_f$ is $2r$, containing both purified and error subspaces, each of rank r. Providing the array manifold errors are small, the eigenvalues of the purified subspace of $\boldsymbol{P}_f$ are about $1-\xi^2$, while the eigenvalues of the error subspace have a mean value of about ξ^2. By performing f/b averaging on a matrix with identical eigenvalues, separation of the purified and error subspaces becomes straightforward.

To demonstrate the effectiveness of subspace domain f/b averaging, it was also applied to the data that produced Figure 2, the results being plotted in Figure 3. The signal estimate variances are 0.0007 and 0.0014 beamwidths respectively. Root-MUSIC with subspace domain f/b averaging performs better than any of the other implementations tried.

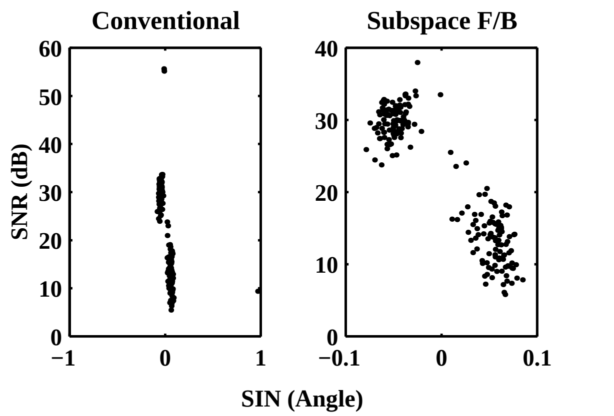

Figure 3: Direction finding results for root-MUSIC with subspace domain f/b averaging for the same data as Figure 2. There are no outling estimates for either of the two signals for the subspace domain f/b averaging.

The implementation changes when the number of signals is larger than half the number of sensors in the array ($r > {}^{N}/_{2}$). In this case only the $N-r$ smallest eigenvectors of $\boldsymbol{P}_f$ are discarded, since there are not enough degrees of freedom to fully represent both the purified and error subspaces. In this case the signal subspace lies slightly less than the 3 dB closer to the presumed array when $r < {}^{N}/_{2}$.

There is not enough room to fully address the effect of coherent signals on subspace domain f/b averaging here. However, if it is known that there are v coherent pairs of signals present, then the largest $r+v$ eigenvectors of $\boldsymbol{P}_f$ are used as the purified subspace.

Figure 4 shows plots of the MUSIC spectrum for a piece of data collected with the Mountaintop experimental radar [9]. There is a single signal source at 30º from array broadside with an SNR of about 30 dB. The array is assumed to be uniform linear. The first eigenvalue of the conventional and f/b covariance matrices is at about 42 dB above the noise floor. For the f/b covariance matrix the second (error) eigenvalue is about 24 dB above the noise floor. This corresponds to $\xi^2 = -15$ dB. The peak using the subspace domain f/b averaging is about 2.5 dB higher, indicating that the subspace lies closer to the presumed array manifold.

5. CONCLUSIONS

Forwards-backwards averaging in the presence of array manifold errors can double the size of the signal subspace resulting in both purified and error signal subspaces. When the purified subspace can be extracted by itself, the DOA estimates are less prone to the effects of array manifold errors. Unfortunately this is not always easily accomplished when f/b averaging is implemented in the data domain. However, the subspace domain forwards-backwards averaging proposed here allows reliable partitioning of the purified and error subspaces, thus reducing algorithm sensitivity to array manifold errors.

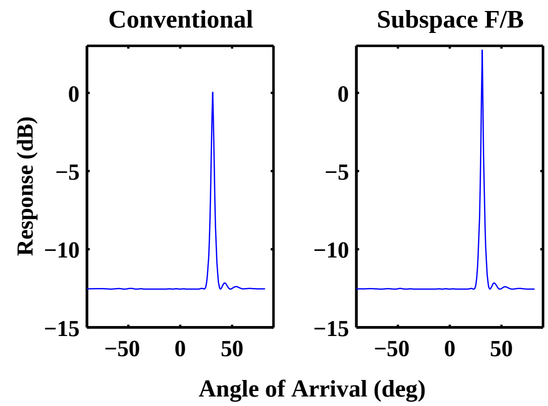

Figure 4: Plots of the MUSIC spectrum with a single signal incident from about 30º on the Mountaintop radar. F/b averaging increases the maximum in the response by about 2.5 dB.

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