

# ANALYTICAL DESIGN OF 3-D WAVELET FILTER BANKS USING THE MULTIVARIATE BERNSTEIN POLYNOMIAL

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*Abstract*— The design of 3-D multirate filter banks where the downsampling/upsampling is on the FCO (Face Centered Orthorhombic) lattice is addressed in this paper. With such a sampling lattice, the ideal 3-D subband of the low-pass filter is of the TRO (TRuncated Octahedron) shape. The transformation of variables has been shown previously to be an effective technique for designing M-D filter banks. We present a design technique for the transformation function using the multivariate Bernstein polynomial which provides good approximation to the TRO subband shape. The method is analytically based and does not require any optimization procedure. Closed form expressions are obtained for the filters of any order. Another advantage of this technique is that it yields filters with a flat frequency response at the aliasing frequency  $(\omega_1, \omega_2, \omega_3) = (\pi, \pi, \pi)$ . The flatness is important for giving regular Discrete Wavelet Transform systems.

## I. INTRODUCTION

Previous work [1] has demonstrated the flexibility and simplicity of the transformation of variables design technique. It is the relative ease in designing the transformation function that makes the technique an effective one. A good approximation to complicated subband shapes such as the TRO (see Figure 1) is achieved. In the previous work [2] the design of the transformation function was based on a combination of windowing and the solution of linear constraint equations. The linear equation approach becomes cumbersome when the size of the transformation is large. In [2] the design of transformation functions of the size of  $7 \times 7 \times 7$  was considered. We present here an alternate approach that uses the multivariate Bernstein polynomial. Explicit analytical expressions are obtained for the transformation function of any size.

Several authors have used the Bernstein polynomial for designing filter and filter banks. Several properties of 1-D FIR filters designed using the Bernstein polynomial are presented in [3]. The design of 1-D orthogonal filter banks is presented in [4]. The design of 2-D diamond half-band filters is presented in [5]. The method in [5] is used in [6] for designing filter banks. The design of 3-D filter banks is also mentioned in [6]. However, the result presented there is not appropriate for the FCO lattice / TRO subband filter bank as it does not

provide the correct approximation to the TRO shape and PR is not achieved.

## II. THEORY

The general transformation of variables design theory can be found in [1]. The formulation for the FCO sampling case can be found in [2] (we refer the reader to this reference for details). In this paper we will only focus on the design of the transformation function which is given by  $M(z_1, z_2, z_3) \equiv \sum_{k_1} \sum_{k_2} \sum_{k_3} m(k_1, k_2, k_3) z_1^{k_1} z_2^{k_2} z_3^{k_3}$  where  $m(k_1, k_2, k_3) = \begin{cases} 0 & \text{for } k_1 + k_2 + k_3 = \text{even} \\ \text{arbitrary} & \text{for } k_1 + k_2 + k_3 = \text{odd} \end{cases}$ . The transformation function is related to a 3-D (TRO) halfband filter:  $H_{HB}(z_1, z_2, z_3) = \frac{1}{2}(1 + M(z_1, z_2, z_3))$ . By deleting the coefficient at the origin of a halfband filter impulse response, we obtain the transformation coefficient, i.e.  $m(k_1, k_2, k_3) = 2h_{HB}(k_1, k_2, k_3) - \delta(k_1, k_2, k_3)$ . In this paper we will use the Bernstein polynomial to design the halfband filter which is subsequently used to obtain the transformation function.

The 3-D Bernstein polynomial [7] is given by

$$B(x, y, z) \equiv \sum_{i=0}^N \sum_{j=0}^N \sum_{k=0}^N f(i, j, k) \binom{N}{i} \binom{N}{j} \binom{N}{k} x^i (1-x)^{N-i} y^j (1-y)^{N-j} z^k (1-z)^{N-k} \quad (1)$$

where  $\binom{N}{i} = \frac{N!}{i!(N-i)!}$  and  $f(i, j, k)$  are the Bernstein coefficients which defines the polynomial. The domain of the function is the unit cube  $C: (x, y, z) \in C \equiv [0, 1]^3$ . The coefficients values are sampled values from an ideal function that the polynomial is trying to approximate, i.e.  $f(i, j, k) = f_I(i/N, j/N, k/N)$ . The function  $f_I(x, y, z)$  is defined over the continuous valued support  $C$ . To obtain the z-transform transfer function,  $H(z_1, z_2, z_3)$ , the following mapping is applied to  $B(x, y, z)$ :  $x = -\frac{1}{4}z_1(1-z_1^{-1})^2$ ,  $y = -\frac{1}{4}z_2(1-z_2^{-1})^2$ ,  $z = -\frac{1}{4}z_3(1-z_3^{-1})^2$ .

If the Bernstein polynomial is to be used for halfband filter design, it must satisfy

$$B(x, y, z) + B(1-x, 1-y, 1-z) = 1. \quad (2)$$

The condition on the Bernstein coefficients are given by the following theorem:

**Theorem 1:** The necessary and sufficient condition for  $B(x, y, z)$  to satisfy the HB condition (2) is:

$$f(i, j, k) + f(N-i, N-j, N-k) = 1. \quad (3)$$

**Proof:** To prove the sufficient part of the theorem, we substitute (1) into the LHS of (2) which yields (after some algebraic manipulation)

$$\begin{aligned} & B(x, y, z) + B(1-x, 1-y, 1-z) \\ &= \sum_{i=0}^N \sum_{j=0}^N \sum_{k=0}^N \{f(i, j, k) + f(N-i, N-j, N-k)\} \\ & \quad \cdot x^i (1-x)^{N-i} y^j (1-y)^{N-j} z^k (1-z)^{N-k}. \end{aligned}$$

Using (3) and the binomial expansion identity  $\sum_{i=0}^N x^i (1-x)^{N-i} = (x+1-x)^N = 1$ , we have proven that LHS = RHS for equation (2).

To prove the necessary part of the theorem, we let  $f(i, j, k) + f(N-i, N-j, N-k) = 1 + p(i, j, k)$ . Substituting (1) into (2) yields the following equation (after simplification):

$$\begin{aligned} & \sum_{i=0}^N \sum_{j=0}^N \sum_{k=0}^N p(i, j, k) x^i (1-x)^{N-i} \\ & \quad y^j (1-y)^{N-j} z^k (1-z)^{N-k} = 0 \end{aligned}$$

This equation must be satisfied for all  $(x, y, z) \in C$ . The only way this is possible is for  $p(i, j, k) = 0 \forall i, j, k$ . Hence (3) is necessary.  $\square$

The degree of flatness of  $B(x, y, z)$  is the same at the points  $(0, 0, 0)$  and  $(1, 1, 1)$  due to symmetry (see (2)). The degree of flatness determines the regularity of the resulting wavelet filters [2] and can be determined by using the following theorem:

**Theorem 2:** Let  $S \geq 0$  (integer). Consider the following constraint:

$$\frac{\partial^{p+q+r} B}{\partial x^p \partial y^q \partial z^r}(0, 0, 0) = 0 \quad (4)$$

for all values of  $p, q$  and  $r$  that satisfy  $p+q+r \leq S$  and  $p, q, r \geq 0$ . The necessary condition for satisfying (4) is:

$$\Delta_x^p \Delta_y^q \Delta_z^r f(0, 0, 0) = 0 \quad \text{for} \quad p+q+r \leq S \quad (5)$$

where  $\Delta_x^p$  is the  $p$ th forward finite difference along the  $x$  direction of  $f$  evaluated at  $(i, j, k) = (0, 0, 0)$ .  $\Delta_y^q$  and  $\Delta_z^r$  are the  $q$ th and  $r$ th difference along the  $y$  and  $z$  directions respectively.

**Proof:** By applying Lemma 6.3.1 in [7] (pg. 112) to each dimension of  $B(x, y, z)$  we have

$$\frac{\partial^{p+q+r} B}{\partial x^p \partial y^q \partial z^r}(0, 0, 0) = C \Delta_x^p \Delta_y^q \Delta_z^r f(0, 0, 0) \quad (6)$$

where  $C > 0$  is a constant. The result follows immediately.  $\square$

### III. BERNSTEIN POLYNOMIAL APPROXIMATION DESIGN

For the TRO subband the plane  $T \equiv \{(x, y, z) : x+y+z = \frac{3}{2}\}$  separates the passband  $P$  and the stopband  $S$  regions.  $T$  is the plane that partitions the unit cube  $C$  into two equal halves ( $P$  and  $S$ ) and it is equidistant to the points  $(0, 0, 0)$  (zero frequency) and  $(1, 1, 1)$  (aliasing frequency). A 3-D grid of discrete points is constructed in  $C$ . Each linear dimension (in the range  $[0, 1]$ ) is uniformly sampled into  $N+1$  points (see (1)). There are  $(N+1)^3$  discrete sampled points in  $C$ . For our designs we shall choose the following ideal function for approximation:

$$f_I(x, y, z) = \begin{cases} 1 & (x, y, z) \in P \\ \frac{1}{2} & (x, y, z) \in T \\ 0 & (x, y, z) \in S \end{cases}.$$

The derivation of the analytical expression will not be presented here for lack of space. There are two cases to consider:

1. When  $N = 2M + 1$  (odd). This is the simpler situation. All the discrete points are either in  $P$  or  $S$ . There are no points on  $T$ . The Bernstein coefficients  $f(i, j, k)$  values are either 1 or 0. The size of the resulting transformation function is  $2N + 1 = 4M + 3$ . The expression for the impulse response is

$$\begin{aligned} & \sum_{k=0}^M \sum_{i=0}^{M-k} \sum_{j=0}^{2M+1-k-i} K(n_1, n_2, n_3; i, j, k) \\ & + \sum_{k=0}^M \sum_{i=M-k+1}^{2M+1} \sum_{j=0}^{3M+1-k-i} K(n_1, n_2, n_3; i, j, k) \\ & + \sum_{k=M+1}^{2M+1} \sum_{i=0}^{3M+1-k} \sum_{j=0}^{3M+1-k-i} K(n_1, n_2, n_3; i, j, k). \quad (7) \end{aligned}$$

2. When  $N = 2M$  (even). This is the more complicated situation as some of the discrete points are on  $T$ . Some of the Bernstein coefficients  $f(i, j, k)$  values are  $\frac{1}{2}$ . The size of the resulting transformation function is  $2N + 1 = 4M + 1$ . The expression for the impulse response is

$$\begin{aligned} & \sum_{k=0}^{M-1} \sum_{i=0}^{M-k-1} \sum_{j=0}^{2M} K(n_1, n_2, n_3; i, j, k) \\ & + \sum_{k=0}^{M-1} \sum_{i=M-k}^{2M} \sum_{j=0}^{3M-k-i-1} K(n_1, n_2, n_3; i, j, k) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=0}^{M-1} \sum_{i=M-k}^{2M} \frac{1}{2} K(n_1, n_2, n_3; i, 3M - k - i, k) \\
 & + \sum_{k=M}^{2M} \sum_{i=0}^{3M-k-1} \sum_{j=0}^{3M-k-i-1} K(n_1, n_2, n_3; i, j, k) \\
 & + \sum_{k=M}^{2M} \sum_{i=0}^{3M-k} \frac{1}{2} K(n_1, n_2, n_3; i, 3M - k - i, k). \quad (8)
 \end{aligned}$$

In both cases it can be verified that condition (3) is satisfied. Both (7) and (8) are defined for  $n_1, n_2, n_3 = 0, 1, \dots, 2N$ . Note that the expressions obtained from the Bernstein polynomial are non-causal zero-phase functions (symmetrical about the origin), but the expressions above are the causal versions (a delay of  $z_1^{-N} z_2^{-N} z_3^{-N}$  is required). The function  $K$  is defined as

$$\begin{aligned}
 K(n_1, n_2, n_3; i, j, k) \equiv & \\
 & \frac{(-1)^{i+j+k}}{4^{3N}} \binom{N}{i} \binom{N}{j} \binom{N}{k} \\
 & \cdot S_N(i, n_1) S_N(j, n_2) S_N(k, n_3)
 \end{aligned}$$

where

$$S_N(x, a) \equiv \sum_{l=\max(0, a-2(N-x))}^{\min(2x, a)} (-1)^l \binom{2x}{l} \binom{2N-2x}{a-l}$$

The next theorem gives the degree of flatness.

**Theorem 3:** The degree of flatness of the filters in (7) and (8) is  $S = N$ .

**Proof:** By extending equation (21) from [3] to 3-D, we have

$$\begin{aligned}
 \Delta_x^p \Delta_y^q \Delta_z^r f(0, 0, 0) = & \\
 & \sum_{i=0}^p \sum_{j=0}^q \sum_{k=0}^r f(i, j, k) (-1)^{p-i} \binom{p}{i} \\
 & \cdot (-1)^{q-j} \binom{q}{j} (-1)^{r-k} \binom{r}{k}. \quad (9)
 \end{aligned}$$

If  $f(i, j, k) = \text{constant}$  in the summation, then by using the binomial identity  $\sum_{i=0}^p (-1)^{p-i} \binom{p}{i} = (1-1)^p = 0$ , it is easily verified that (9) is zero. Now for the filters in (7) and (8),  $f(i, j, k) = 1$  for  $(i, j, k) \in P_D$ , where  $P_D$  is the set of indices representing the discrete points in  $P$ . The indices in  $P_D$  can be obtained from the summation indices in (7) and (8) (the third and fifth sums in (8) which has the factor  $\frac{1}{2}$  is excluded). The region  $P$  is bounded by the horizontal/vertical planes  $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$  and also by the oblique plane  $T$ . The planes  $x = 1, y = 1$  and  $z = 1$  (corresponding to  $i = N, j = N$  and  $k = N$ ) respectively limit the values of  $p, q$  and  $r$  to  $N$  (in order for (9) to be zero). Hence by Theorem 2, (4) is satisfied for  $S = N$ .  $\square$

We shall consider slices of the 3-D frequency response to give 2-D frequency response plots. The slices we consider are across the plane  $\omega_3 = \text{constant}$ . The shape of the ideal 2-D slice is shown in Figure 2. Ideally the transformation has values 1 and  $-1$  in the passband and stopband respectively.

*Example 1:*  $N = 4$ . Formula (8) gives a transformation of size  $9 \times 9 \times 9$ . The frequency response is shown in Figure 3.

*Example 2:*  $N = 7$ . Formula (7) gives a transformation of size  $15 \times 15 \times 15$ . The frequency response is shown in Figure 4.

The responses start off with the value 1 in the passband and eventually falls to the value  $-1$  in the stopband as required by the specification. As the size of the transformation increases, both the sharpness of roll-off and the degree of flatness increase. Finally, note that with  $N = 3$ , formula (7) gives the transformation that is exactly the same as the transformation in Example 6.1 in [2]. The results here; and the comparison made with the result in [2] verifies the formulas (7) and (8).

## V. CONCLUSIONS

The multivariate Bernstein polynomial has provided an easy and effective way of designing 3-D FCO/TRO wavelet filter banks. Closed form analytical expressions were obtained for designing the transformation function of any size and no optimization is required. Good approximation to the TRO subband shape is achieved. Arbitrarily flat frequency response is obtainable for giving regular Discrete Wavelet Transform systems.

## REFERENCES

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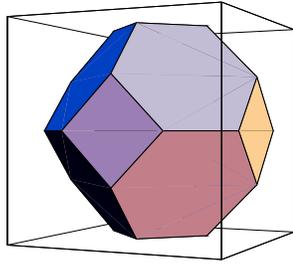


Fig. 1. Ideal passband with FCO sampling lattice: *Truncated Octahedron*. Also shown is the unit frequency cell in 3D

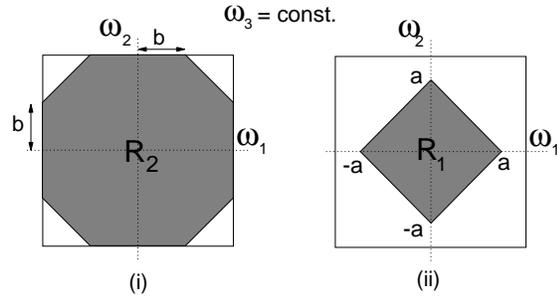


Fig. 2. Slice across the plane  $\omega_3 = \text{const.}$  for the TRO. (i)  $0 \leq |\omega_3| < \pi/2$  and  $b = \pi/2 - |\omega_3|$ . (ii)  $\pi/2 \leq |\omega_3| \leq \pi$  and  $a = 3\pi/2 - |\omega_3|$ .

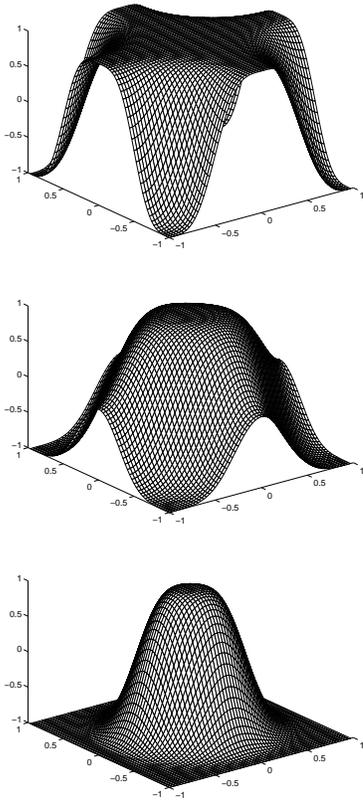


Fig. 3. 2D slices of the 3D frequency in Example 1:  $N = 4$ . Transformation size is  $9 \times 9 \times 9$ . Top: slice across  $\omega_3 = 0$ . Middle:  $\omega_3 = \pi/2$ . Bottom:  $\omega_3 = \pi$ .

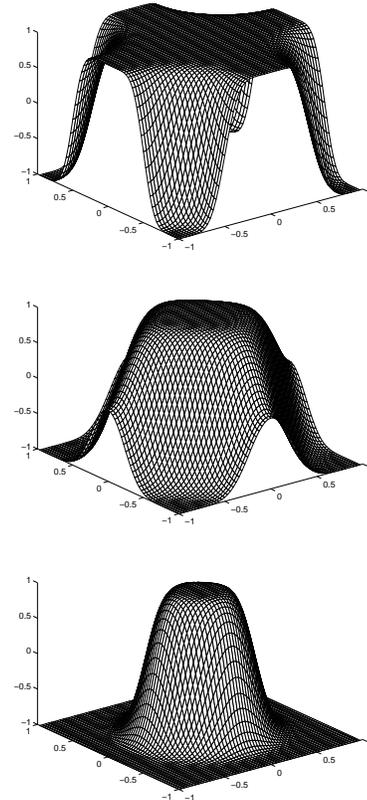


Fig. 4. 2D slices of the 3D frequency in Example 2:  $N = 7$ . Transformation size is  $15 \times 15 \times 15$ . Top: slice across  $\omega_3 = 0$ . Middle:  $\omega_3 = \pi/2$ . Bottom:  $\omega_3 = \pi$ .