COMPARISON OF THE THEORETICAL PERFORMANCE BOUNDS FOR TWO WAVEFRONT CURVATURE RANGING TECHNIQUES

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ABSTRACT

The Range Focused Beamformer (RFB) and the Triple Aperture Crosscorrelator (TAC) are the two primary wavefront curvature ranging techniques used in fielded sonar systems. Theoretical performance bounds have been presented in the past for both approaches. This paper develops unified array processing performance bounds where the RFB and the TAC are special cases. Specifically, general Cramer-Rao Lower Bounds (CRLB) on range and bearing estimation for a linear array of directional elements are developed, where the CRLB for the RFB and the TAC are shown to be special cases of the general theory. The ranging performance of the two techniques are then compared.

1. INTRODUCTION

Joint passive estimation of localization parameters (range and bearing) using a long towed array is an ongoing problem of considerable interest. When sources are close to the array, the arriving acoustic wavefront cannot be assumed to be planar. In general, in the near-field the wavefront is spherical, in the Fresnel region the wavefront is cylindrical (quadric) and in the Fraunhofer (far-field) region it is planar. For near-field ranging, the Range-Focused Beamformer (RFB) and the Triple-Aperture Array (TAC) are the two primary candidate system architectures for fieldable systems (See Figures 1 and 2). Performance of the range and bearing estimation process for each system can be quantified by the Cramer-Rao Lower Bound (CRLB). Although there are numerous papers in the literature devoted to the development of analytical performance bounds for either architecture [1] - [6], the authors are unaware of any paper that present a side-by-side analytical performance comparison for the two systems. Such a side-by-side comparison will be presented below.

2. GENERAL THEORY

2.1 Signal and Noise Models

In order to include both the RFB and the TAC architectures, consider a linear array of M elements, where the elements themselves can be beamformed subarrays. The output of the m$^{\text{th}}$ element is given by

$$x_m(t) = s_m(t) + n_m(t) \tag{1}$$

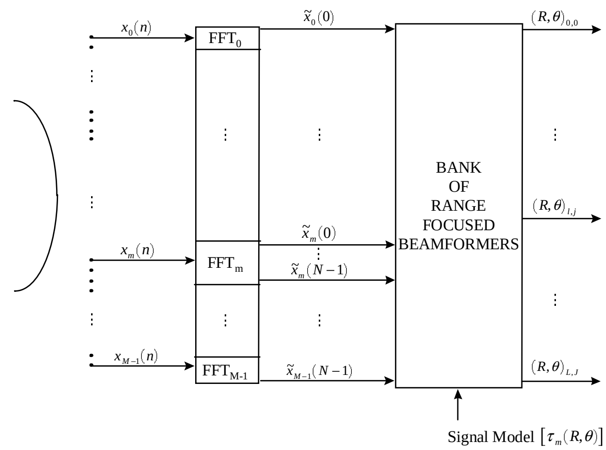

Figure 1. Structure of a range focused beamformer

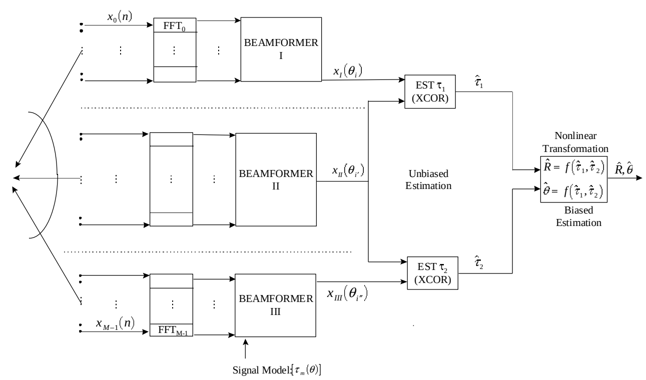

Figure 2. Structure of a Triple Aperture Crosscorrelator

were the signal and noise components, $s_m(t)$ and $n_m(t)$ respectively, are zero-mean real-valued uncorrelated Gaussian random processes with unknown spectra $S_s(f)$ and $S_n(f)$. The data is linearly transformed by Fourier transforming the time series over a coherent time block of T seconds resulting in a set of random positive frequency components $\tilde{x}_m(f_1), \tilde{x}_m(f_2), \cdots, \tilde{x}_m(f_N)$. This set is described by a complex jointly Gaussian distribution with negligible correlation between frequencies for sufficiently large time-bandwidth products.

Define the vector sets

$$\vec{\tilde{\mathbf{x}}}(f_k) \equiv \left\{\tilde{x}_0(f_k), \tilde{x}_1(f_k), \cdots, \tilde{x}_{M-1}(f_k)\right\}^T \tag{2}$$

$$\vec{\mathbf{x}} \equiv \left\{ \tilde{\vec{\mathbf{x}}}(f_1), \tilde{\vec{\mathbf{x}}}(f_2), \cdots, \tilde{\vec{\mathbf{x}}}(f_N) \right\}^T \tag{3}$$

Define similar vector sets $\tilde{\vec{\mathbf{s}}}(f_k)$, $\vec{\mathbf{s}}$, $\tilde{\vec{\mathbf{n}}}(f_k)$, and $\vec{\mathbf{n}}$. Assuming that the signal wavefront crosses the m[th] element with a time delay τ_m relative to a reference element, the signal component at frequency f_k is given by

$$\tilde{s}_m(f_k) = S(f_k) e^{-j\omega_k \tau_m} \tag{4}$$

where $\omega_k = 2\pi f_k$. Thus

$$\tilde{\vec{\mathbf{s}}}(f_k) = S(f_k)\vec{\mathbf{d}}(f_k) \tag{5}$$

with

$$\vec{\mathbf{d}}(f_k) \equiv \left\{ e^{-j\omega_k \tau_0}, e^{-j\omega_k \tau_1}, \cdots, e^{-j\omega_k \tau_{M-1}} \right\}^T \tag{6}$$

Two points must be made concerning the signal model. First, if the array elements are beamformed subarray outputs, then the beam chosen for each subarray is the beam containing the signal. It is also assumed that the signal is near the maximum response axis of that beamformer channel, where the relative signal gain is unity.

The second point concerns selection of the reference element. In RFB, element zero is usually chosen as the reference, while historically, the center element is chosen for the TAC. Mathematically, it does not effect the performance bounds, so for convenience, the RFB convention will be chosen. Thus, τ_m is the time delay relative to the 0[th] element and $\tau_0 = 0$.

The signal correlation matrix at frequency f_k is defined by

$$\Gamma_s(f_k) = E\left\{ \tilde{\vec{\mathbf{s}}}(f_k) \tilde{\vec{\mathbf{s}}}^H(f_k) \right\} \tag{7}$$

where $\tilde{\vec{\mathbf{s}}}^H(f_k)$ is the complex conjugate transpose of $\tilde{\vec{\mathbf{s}}}(f_k)$.

Using Eqs. (4) and (5), this becomes

$$\Gamma_s(f_k) = TS_s(f_k)\vec{\mathbf{d}}(f_k)\vec{\mathbf{d}}^H(f_k) \tag{8}$$

where

$$TS_s(f_k) \equiv E\left\{ S(f_k) S^*(f_k) \right\} \tag{9}$$

and, since the signal components are approximately uncorrelated across frequencies, the total signal correlation matrix is block diagonal:

$$\Gamma_s \equiv T \cdot diag\left\{ \Gamma_s(f_1), \Gamma_s(f_2), \cdots, \Gamma_s(f_N) \right\}^T \tag{10}$$

The noise component at frequency f_k is defined by

$$\tilde{n}_m(f_k) = q_m(f_k) N(f_k) \tag{11}$$

where $q_m(f_k)$ is a noise weighting factor for the m[th] element. This factor allows us to include the effective array gain for the subarrays of the TAC and to take into account spatial noise correlation.

The noise correlation matrix at frequency f_k is defined by

$$\Gamma_n(f_k) \equiv TS_n(f_k)\vec{\mathbf{q}}(f_k)\vec{\mathbf{q}}^H(f_k) \tag{12}$$

where

$$\vec{\mathbf{q}}(f_k) \equiv \left\{ q_0(f_k), q_1(f_k), \cdots, q_{M-1}(f_k) \right\} \tag{13}$$

and

$$TS_n(f_k) \equiv E\left\{ N(f_k) N^*(f_k) \right\} \tag{14}$$

The total noise correlation matrix is block diagonal

$$\Gamma_n \equiv T \cdot diag\left\{ \Gamma_n(f_1), \Gamma_n(f_2), \cdots, \Gamma_n(f_N) \right\}^T \tag{15}$$

And the total signal plus noise correlation matrix is

$$\Gamma = \Gamma_s + \Gamma_n \tag{16}$$

2.2 Cramer-Rao Lower Bounds

Let the parameter $\vec{\theta}$ denote the vector of parameters to be estimated, i.e. $\vec{\theta} = \left\{\theta_1, \theta_2\right\}^T = \left\{R, \beta\right\}^T$. Under complex Gaussian signal and noise assumptions, the probability density function that represents the likelihood of a given $\vec{\theta}$ for a given set of measurements is described by the Likelihood function

$$p\left(\vec{\mathbf{x}} | \vec{\theta}\right) = \frac{\exp\left(-\vec{\mathbf{x}}^H \Gamma^{-1} \vec{\mathbf{x}}\right)}{\pi^{NM} |\Gamma|} \tag{17}$$

For an unbiased estimator, $\hat{\vec{\theta}}$, a lower bound on the variance of the estimate is given by the Cramer-Rao Lower Bound (CRLB)[7]. The CRLB for the components of $\hat{\vec{\theta}}$ is

$$\Psi_{CRLB} = \begin{bmatrix} \sigma_R^2 & \mu_{R\beta} \\ \mu_{\beta R} & \sigma_\beta^2 \end{bmatrix} = \mathbf{J}_{\vec{\theta}}^{-1}, \tag{18}$$

where $\mathbf{J}_{\vec{\theta}}$ is the Fisher Information Matrix (FIM) defined by

$$\mathbf{J}_{\vec{\theta}} = -E\left\{ \vec{\nabla}_{\vec{\theta}} \vec{\nabla}_{\vec{\theta}}^T l\left(\vec{\mathbf{x}} | \vec{\theta}\right) \right\} \tag{19}$$

with

$$\vec{\nabla}_{\vec{\theta}} = \left\{ \frac{\partial}{\partial R}, \frac{\partial}{\partial \beta} \right\}^T \tag{20}$$

The log-likelihood function is then given by

$$l\left(\vec{\mathbf{x}} | \vec{\theta}\right) = \ln\left[p\left(\vec{\mathbf{x}} | \vec{\theta}\right) \right] = -NM\ln(\pi) - \ln|\Gamma| - \vec{\mathbf{x}}^H \Gamma^{-1} \vec{\mathbf{x}} \tag{21}$$

Typically, a number of observation intervals are averaged during the estimation process. Let the frequency variable f_k become $f_{k,i}$, where the index i refurs to the scan number. If the data is statistically independent between observation intervals, then the cumulative log-likelihood function is simply

$$l\left(\vec{\mathbf{x}} | \vec{\theta}\right) = \sum_{i=1}^{I} l\left(\vec{\mathbf{x}}_i | \vec{\theta}\right) = \sum_{I=1}^{I} \sum_{k=1}^{N} l_T\left(f_{k,i}\right) + \text{constant} \tag{22}$$

where

$$l_T(f_{k,i}) = -\ln\left|\Gamma(f_{k,i})\right| - \vec{\mathbf{x}}^H(f_{k,i})\Gamma^{-1}(f_{k,i})\vec{\mathbf{x}}(f_{k,i}) \quad (23)$$

Thus, Eq. (19) can be rewritten as

$$\mathbf{J}_{\vec{\theta}} = \sum_{i=1}^{I}\sum_{k=1}^{N}\mathbf{J}_{\vec{\theta}}(f_{k,i}) \quad (24)$$

with

$$\mathbf{J}_{\vec{\theta}}(f_{k,i}) = -E\left\{\vec{\nabla}_{\vec{\theta}}\vec{\nabla}_{\vec{\theta}}^T l_T(f_{k,i})\right\} \quad (25)$$

Noting that from Eq. (6), (8), (16) and (23), $\Gamma(f_{k,i})$ depends only on $\vec{\tau} = \{\tau_1, \tau_2, \cdots, \tau_{M-1}\}$ and that $\vec{\tau}$ is a function of $\vec{\theta}$, then the chain rule can be used to obtain

$$\mathbf{J}_{\vec{\theta}}(f_{k,i}) = \mathbf{U}^T\mathbf{J}_{\vec{\tau}}(f_{k,i})\mathbf{U} \quad (26)$$

where

$$\mathbf{J}_{\vec{\tau}}(f_{k,i}) = -E\left\{\vec{\nabla}_{\vec{\tau}}\vec{\nabla}_{\vec{\tau}}^T l_T(f_{k,i})\right\} \quad (27)$$

$$\mathbf{U} = \begin{bmatrix} \frac{\partial\tau_1}{\partial R} & \frac{\partial\tau_2}{\partial R} & \cdots & \frac{\partial\tau_{M-1}}{\partial R} \\ \frac{\partial\tau_1}{\partial\beta} & \frac{\partial\tau_2}{\partial\beta} & \cdots & \frac{\partial\tau_{M-1}}{\partial\beta} \end{bmatrix}^T \quad (28)$$

$$\vec{\nabla}_{\vec{\tau}} = \left\{\frac{\partial}{\partial\tau_1}, \frac{\partial}{\partial\tau_2}, \cdots, \frac{\partial}{\partial\tau_{M-1}}\right\}^T \quad (29)$$

The CRLB for the time delay estimation process are given by:

$$\Psi_{\vec{\tau}}^{CRLB} = \begin{bmatrix} \sigma_{\tau_1}^2 & \mu_{\tau_1\tau_2} & \cdots & \mu_{\tau_1\tau_{M-1}} \\ \mu_{\tau_2\tau_1} & \sigma_{\tau_2}^2 & & \\ \vdots & & \ddots & \\ \mu_{\tau_{M-1}\tau_2} & & & \sigma_{\tau_{M-1}}^2 \end{bmatrix} = \mathbf{J}_{\vec{\tau}}^{-1} \quad (30)$$

3. SPECIAL CASES

3.1 Bounds on Time Delay Estimation

Assume that the noise is element-to-element independent. Then,

$$q_m(f_{k,i})q_{m'}(f_{k,i}) \equiv w_m(f_{k,i})\delta_{mm'} \quad (31)$$

Using the Woodbury matrix inversion lemma to determine $\Gamma^{-1}(f_{k,i})$, Eq. (27) yields

$$\mathbf{J}_{\vec{\tau}}(f_{k,i}) = 2\omega_k^2 F^2(f_{k,i})\mathbf{P}(f_{k,i}), \quad (32)$$

where

$$F^2(f_{k,i}) = \frac{S_s^2(f_{k,i})/S_n^2(f_{k,i})}{1 + \frac{S_s(f_{k,i})}{S_n(f_{k,i})}\sum_{m=0}^{M-1} w_m^{-1}(f_{k,i})}, \quad (33)$$

$$\mathbf{P}(f_{k,i}) = \left[\sum_{m=0}^{M-1} w_m^{-1}(f_{k,i})\right]\mathbf{Q}^{-1}(f_{k,i}) - \vec{\mathbf{w}}(f_{k,i})\vec{\mathbf{w}}^T(f_{k,i}), \quad (34)$$

$$\vec{\mathbf{w}}(f_{k,i}) = \left\{w_1^{-1}(f_{k,i}), w_2^{-1}(f_{k,i}),, w_{M-1}^{-1}(f_{k,i})\right\}^T, \quad (35)$$

and

$$\mathbf{Q}(f_{k,i}) = \mathbf{I}\cdot\vec{\mathbf{w}}^T(f_{k,i}). \quad (36)$$

Note that $\mathbf{I}$ is a unit diagonal matrix of dimension $(M-1)\times(M-1)$.

For the RFB, each element is omnidirectional, hence $w_m(f_{k,i}) = 1$ for all m. Thus, Eq. (32) yields

$$\mathbf{J}_{\vec{\tau}}^{RFB}(f_{k,i}) = 2\omega_k^2\frac{S_s^2(f_{k,i})/S_n^2(f_{k,i})}{1 + M\frac{S_s(f_{k,i})}{S_n(f_{k,i})}}\begin{pmatrix} M-1 & -1 & \cdots & -1 \\ -1 & M-1 & & \\ \vdots & & \ddots & \\ -1 & & & M-1 \end{pmatrix} \quad (37)$$

To facilitate comparison of the RFB with the TAC, we set the <u>total</u> number of hydrophones in both systems to M. For the TAC, assume that the array consists of three plane wave beam-formed subarrays with $M/4$ elements in the two outer subarrays and $M/2$ elements in the center subarray [2]. For this case, $w_m(f_{k,i})$ is the inverse of the array noise gain for the m[th] subarray at the subarray design frequency. Thus, $w_0(f_{k,i}) = w_2(f_{k,i}) = 4/M$ and $w_1(f_{k,i}) = 2/M$, and Eq. (32) becomes

$$\mathbf{J}_{\vec{\tau}}^{TAC}(f_{k,i}) = 2\omega_k^2\frac{S_s^2(f_{k,i})/S_n^2(f_{k,i})}{1 + M\frac{S_s(f_{k,i})}{S_n(f_{k,i})}}\frac{1}{16}M^2\begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix} \quad (38)$$

3.2 Bounds on Range andBearing Estimation

If R is the range to the acoustic source measured from the array center, β is the angle measured clockwise from broadside, d is the element spacing and c is the speed of sound in water, then the relative time delay associated with the m[th] element can be approximated to second order by

$$\tau_m(R,\theta) = m\frac{d}{c}\left[\sin\beta + \frac{1}{2}\frac{d}{R}a_1(m)\cos^2\beta + \frac{1}{2}\left(\frac{d}{R}\right)^2 a_2(m)\sin\beta\cos^2\beta\right] \quad (39)$$

where

$$a_1(m) = m - (M-1) \quad (40)$$

$$a_2(m) = m^2 - \frac{3}{2}m(M-1) + \frac{3}{4}(M-1)^2 \quad (41)$$

After some straightforward algebra, using Eqs. (26), (28), (37) and (39), the elements of $\mathbf{J}_{\vec{\theta}}^{RFB}(f_{k,i})$ are given by

$$J_{\theta_i\theta_j}^{RFB}(f_{k,i}) = G_\tau\left[M\sum_{m=1}^{M-1}\frac{\partial\tau_m}{\partial\theta_i}\frac{\partial\tau_m}{\partial\theta_j} - \sum_{m=1}^{M-1}\frac{\partial\tau_m}{\partial\theta_i}\sum_{m=1}^{M-1}\frac{\partial\tau_m}{\partial\theta_j}\right] \quad (42)$$

with

$$G_\tau = 2\sum_{i=1}^{I}\sum_{k=1}^{N}\omega_k^2 \frac{S_s^2\left(f_{k,i}\right)/S_n^2\left(f_{k,i}\right)}{1+M\dfrac{S_s\left(f_{k,i}\right)}{S_n\left(f_{k,i}\right)}} \tag{43}$$

The CRLB is then obtained from Eq. (18). See [8] for similar results for matched field processing.

For the TAC, R and β are estimated by taking the terms for τ_1 and τ_2 only up to first order in Eq. (39) and inverting to obtain R and β. This results in biased estimates, with the ignored second-order term acting as the bias [6]. Using the CRLB for biased estimators [6][7], the diagonal elements of Eq. (18) for the TAC are given by

$$\sigma_R^2 = \left[1+3\frac{\sigma_{RU}^2}{R^2}\right]^2 \sigma_{RU}^2, \tag{44}$$

where

$$\sigma_{RU}^2 = \left[\frac{cR^2}{L^2\cos^2\beta}\right]^2 \left(\sigma_{\tau_1}^2 + \sigma_{\tau_2}^2 + 2\mu_{\tau_1\tau_2}\right), \tag{45}$$

and

$$\sigma_\beta^2 = \left[1+\frac{1}{2}\sigma_{\beta U}^2\left(1+3\tan^2\beta\right)\right]^2 \sigma_{\beta U}^2, \tag{46}$$

where

$$\sigma_{\beta U}^2 = \left[\frac{c}{2L\cos\beta}\right]^2 \left(\sigma_{\tau_1}^2 + \sigma_{\tau_2}^2 - 2\mu_{\tau_1\tau_2}\right). \tag{47}$$

where the elements of CRLB for time delay estimation are obtained by inverting Eq. (38).

4. RANGING COMPARISON

Assume that the processed frequency band is the octave whose upper band limit is the design frequency of the RFB array $\left(f_{des} = c/2d\right)$. Also assume that both the signal and noise have a constant spectrum over the band, and that the in-band signal-to-noise ratio is -20 dB at the input to the hydrophones. For this case, Figure 3 shows a comparison of the ranging performance of both the RFB and the TAC in ranges relative to the reference array length. The reference array length is taken as that of the RFB array, L_{RFB} . When the TAC is the same length as the RFB (i.e. the outer subarrays of the TAC are adjacent to the inner subarray), Figure 3 (a) shows that ranging errors for the TAC are about a factor of 10 higher than those of the RFB. Figure 3 (b) shows that when the TAC array is twice as long as the RFB array (i.e. the outer subarrays of the TAC are moved out so that the total array length is twice that of the RFB), it's performance is comparable to that of the RFB array.

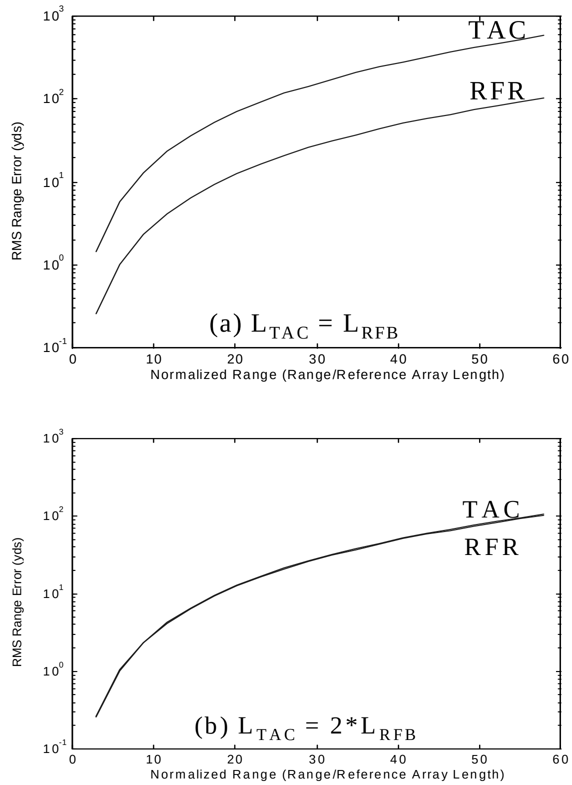

Figure 3. RFB versus TAC ranging perfomance.

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