

A POLYNOMIAL ROOTING APPROACH TO THE LOCALIZATION OF COHERENTLY SCATTERED SOURCES

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ABSTRACT

The problem of passive localization of coherently scattered sources with an array of sensors is considered. The spatial extent of such a source is typically characterized by an angular mean and an angular spreading parameter. The maximum likelihood (ML) estimator for this problem requires a complicated search of dimension equal to twice the number of sources. However, a previously reported sub-optimal MUSIC type method reduces the search dimension to two (independently of the number of sources). In this paper, the search over the angular mean parameter in the above MUSIC type technique is replaced by a possibly more efficient polynomial rooting procedure. Computer simulations verify the effectiveness of the proposed method compared to the performance of the ML and MUSIC estimators as well as to the Cramer-Rao Bound.

1. INTRODUCTION

Most passive source localization research over the past several years has focused on the estimation of the spatial parameters of sources which are modeled as single points in space. However, in applications such as mobile communications and sonar where multipath or scattering effects are present, a so-called “scattered” source model may be more appropriate. A scattered source can be thought of as possessing spatial extent over some continuum of directions. This spatial extent is typically characterized by a parametric spatial density function. As in [1], it is convenient to distinguish between “coherently distributed” and “incoherently distributed” scattered sources. The received signal components due to a coherently (incoherently) distributed source are correlated (uncorrelated) at each direction over which the scattering extends.

The localization problem for scattered sources is one of using the data received by an array of passive sensors to estimate the parameters of each sources’ spatial density function. Maximum likelihood (ML) approaches to the problem (e.g., [2], [3]) involve a multi-dimensional search whose dimension increases as the number of sources increases. The resulting high computational complexity has led to the development of a more economical sub-optimal,

sub-space type approach based on the MUSIC algorithm (e.g., [1], [4]).

In this paper, a modification of the above subspace approach is presented for coherently distributed scattered sources. In particular, the search over the mean angle parameter of the spatial density function is replaced by a polynomial rooting procedure. For multi-modal cost functions, polynomial rooting may be a computationally attractive alternative to an exhaustive search. The procedure is similar in spirit to those taken for multi-parameter localization of near field sources [5] and sources in shallow water [6].

The paper is organized as follows: The specific problem to be solved is formulated in Section 2. The ML and MUSIC approaches to the problem are reviewed in Section 3. The new polynomial rooting method is described in Section 4. Performance results based on computer simulations are presented in Section 5. Lastly, some conclusions and directions for future work are noted in Section 6.

2. PROBLEM FORMULATION

Consider N narrow band, far field scattered sources whose wavefields impinge on a co-planar array of M passive sensors. The received snapshot of sensor outputs at time t can be modeled as [1]:

$$\mathbf{z}(t) = \sum_{n=1}^N \int_{-\infty}^{\infty} \mathbf{a}(\theta) s_n(\theta, t | \boldsymbol{\psi}_n) d\theta + \mathbf{w}(t) \quad (1)$$

$$\mathbf{a}(\theta) = [e^{-j\omega_o \tau_1(\theta)}, \dots, e^{-j\omega_o \tau_M(\theta)}]^T \quad (2)$$

where $[\cdot]^T$ denotes the vector transpose operation, $\mathbf{a}(\theta)$ is the standard $M \times 1$ far-field point source steering vector at direction of arrival (DOA) and center frequency, θ , and ω_o , respectively, and $s_n(\theta, t | \boldsymbol{\psi}_n)$ is the so-called “spatio-temporal signal density” of the n th source whose spatial component is characterized by a spatial parameter vector $\boldsymbol{\psi}_n$. The time for the signal emitted by a source at θ to travel from the array origin to the m th sensor is given by:

$$\tau_m(\theta) = -\frac{1}{c}(y_{1m} \cos \theta + y_{2m} \sin \theta), \quad (3)$$
$$m \in \{1, \dots, M\}$$

where $\{y_{1m}, y_{2m}\}$ and c denote respectively the Cartesian coordinates specifying the location of the m th sensor and

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the speed of propagation. Lastly, $\mathbf{w}(t)$ is a vector of additive spatio-temporally white noise of variance σ^2 .

For coherently distributed scattered sources, the spatio-temporal signal density function can be decomposed into a temporal component and a deterministic spatial density function parameterized by its spatial parameter vector:

$$s_n(\theta, t | \boldsymbol{\psi}_n) = p(\theta | \boldsymbol{\psi}_n) u_n(t), \quad n \in \{1, \dots, N\} \quad (4)$$

This enables us to rewrite (1) as:

$$\mathbf{z}(t) = \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t) \quad (5)$$

$$\begin{aligned} \mathbf{B} &= [\mathbf{b}(\boldsymbol{\psi}_1), \dots, \mathbf{b}(\boldsymbol{\psi}_N)] \\ \mathbf{b}(\boldsymbol{\psi}_n) &= \int_{-\infty}^{\infty} \mathbf{a}(\theta) p(\theta | \boldsymbol{\psi}_n) d\theta \\ \mathbf{u}(t) &= [u_1(t), \dots, u_N(t)]^T \end{aligned}$$

The spatial density functions considered in this paper are assumed to be completely characterized by two parameters: a mean angle parameter $\psi_1 \in [-\pi, \pi)$ and an angular spreading parameter, ψ_2 .

Note that if the signal waveforms, $\{u_n(t)\}_{n=1}^N$, are stationary zero mean random processes, then the array snapshot is a stationary zero mean random vector process with covariance:

$$\begin{aligned} \mathbf{R} &= E[\mathbf{z}(t)\mathbf{z}^H(t)] = \mathbf{B}\boldsymbol{\Gamma}\mathbf{B}^H + \sigma^2\mathbf{I} \\ \boldsymbol{\Gamma} &= E[\mathbf{u}(t)\mathbf{u}^H(t)] \end{aligned} \quad (6)$$

where $[\cdot]^H$ denotes the conjugate transpose operation.

The spatial parameter estimation problem for coherently distributed scattered sources is one of estimating the N spatial parameter vectors $\{\boldsymbol{\psi}_n\}_{n=1}^N$ from the received data. It is assumed that the spatial density function type is the same for all N sources and is completely known to within its spatial parameter vector. Also, the number of sources is assumed known and less than the number of sensors: $N < M$.

3. ML AND MUSIC ESTIMATORS

In this section, the ML and MUSIC estimators for $\{\boldsymbol{\psi}_n\}_{n=1}^N$ are presented. Let us assume that L discrete time samples of the snapshot vector $\{\mathbf{z}(l)\}_{l=1}^L$ are available. Beginning with the ML estimator, if the signal and noise waveforms are Gaussian, the snapshot vector itself is Gaussian. As in the point source case (e.g., [7]), the resulting estimator requires a search over $N^2 + 2N + 1$ parameters. However, again as in the point source case, ML estimates of the signal covariance matrix and noise power can be formed in terms of the spatial parameters allowing the search dimension to be reduced to $2N$ (see [3]). Such an estimator is written as:

$$\begin{aligned} \hat{\boldsymbol{\zeta}}_{RML} &= \arg \min_{\boldsymbol{\zeta}} Q_{RML}(\boldsymbol{\zeta}), \\ Q_{RML}(\boldsymbol{\zeta}) &= \left[\ln |\mathbf{B}\hat{\boldsymbol{\Gamma}}\mathbf{B}^H + \hat{\sigma}^2\mathbf{I}| \right] \\ \hat{\boldsymbol{\Gamma}} &= \mathbf{B}^\#(\hat{\mathbf{R}} - \hat{\sigma}^2\mathbf{I})\mathbf{B}^{\#H} \end{aligned} \quad (7)$$

$$\begin{aligned} \hat{\sigma}^2 &= \frac{1}{M-N} \text{tr} \left(\mathbf{P}_B^\perp \hat{\mathbf{R}} \right) \\ \hat{\mathbf{R}} &= \frac{1}{L} \sum_{l=1}^L \mathbf{z}(l)\mathbf{z}^H(l) \\ \mathbf{B}^\# &= (\mathbf{B}^H\mathbf{B})^{-1}\mathbf{B}^H, \quad \mathbf{P}_B^\perp = \mathbf{I} - \mathbf{B}\mathbf{B}^\# \\ \boldsymbol{\zeta} &= [\psi_1^T, \dots, \psi_N^T]^T \end{aligned} \quad (8)$$

provided $\hat{\boldsymbol{\Gamma}}$ is positive semi-definite where $\ln(\cdot)$ and $|\cdot|$ respectively denote the natural logarithm and matrix determinant operation.

Assuming that the sources are not perfectly correlated, the MUSIC estimator exploits the orthogonality between the N spatial signature vectors and the $M-N$ "noise subspace" eigenvectors associated with the $M-N$ smallest eigenvalues of the covariance matrix of (6). In practice, the noise subspace eigenvectors from the estimated covariance matrix of (8) are computed, and the estimates are given as the N lowest minima of the MUSIC cost function:

$$\hat{\boldsymbol{\zeta}}_{MU} = \arg N \text{ lowest } \min_{\boldsymbol{\zeta}} Q_{MU}(\psi_1, \psi_2) \quad (9)$$

$$\begin{aligned} Q_{MU}(\psi_1, \psi_2) &= \left[\frac{\mathbf{b}^H \hat{\mathbf{E}}_n \hat{\mathbf{E}}_n^H \mathbf{b}}{\mathbf{b}^H \mathbf{b}} \right] \\ \hat{\mathbf{R}} &= \sum_{m=1}^M \hat{\lambda}_m \hat{\mathbf{e}}_m \hat{\mathbf{e}}_m^H, \quad \hat{\mathbf{E}}_n = [\hat{\mathbf{e}}_{N+1}, \dots, \hat{\mathbf{e}}_M] \\ \hat{\lambda}_1 &\geq \dots \geq \hat{\lambda}_M \end{aligned}$$

where $\{\hat{\lambda}_m\}_{m=1}^M$ and $\{\hat{\mathbf{e}}_m\}_{m=1}^M$ respectively denote the eigenvalues and eigenvectors of the estimated covariance matrix.

4. POLYNOMIAL ROOTING APPROACH

Consider the form of the spatial signature vector, $\mathbf{b}(\psi_1, \psi_2)$. As in [5]-[6], the polynomial rooting modification to the MUSIC algorithm is motivated by observing that each element of the spatial signature vector can be expressed as a Fourier series in the mean angle parameter, ψ_1 :

$$\begin{aligned} b_m(\psi_1, \psi_2) &= \sum_{k=-\infty}^{\infty} C_{mk}(\psi_2) e^{jk\psi_1}, \\ C_{mk}(\psi_2) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} b_m(\psi_1, \psi_2) e^{-jk\psi_1} d\psi_1 \\ m &\in \{1, \dots, M\} \end{aligned} \quad (10)$$

where $b_m = [\mathbf{b}]_m$ denotes the m th element of the spatial signature vector. In practice, (10) is often well approximated by a $2K+1$ term truncated Fourier series:

$$b_m(\psi_1, \psi_2) \approx \sum_{k=-K}^K C_{mk}(\psi_2) e^{jk\psi_1} \quad (11)$$

which in matrix form can be expressed as:

$$\begin{aligned} \mathbf{b}(\psi_1, \psi_2) &\approx \mathbf{C}(\psi_2)\mathbf{g}(\psi_1) \\ [\mathbf{C}(\psi_2)]_{mk} &= C_{m, k-K+1}(\psi_2), \quad [\mathbf{g}(\psi_1)]_k = e^{j(k-K+1)\psi_1} \\ k &\in \{1, \dots, 2K+1\} \end{aligned} \quad (12)$$

where $\mathbf{C}(\psi_2)$ is a $M \times (2K+1)$ matrix of Fourier coefficients and $\mathbf{g}(\psi_1)$ is a $(2K+1) \times 1$ polynomial type vector of complex exponentials. The approximation in (12) motivates the definition of a new cost function based on (9):

$$\bar{Q}_{PR}(z, \psi_2) = \left[\mathbf{g}(1/z) \mathbf{C}^H(\psi_2) \hat{\mathbf{E}}_n \hat{\mathbf{E}}_n^H \mathbf{C}(\psi_2) \mathbf{g}(z) \right] \quad (13)$$

$$[\mathbf{g}(z)]_k = z^{k-K-1}, \quad z = e^{j\psi_1}.$$

Note that $\bar{Q}_{PR}(z, \psi_2)$ is a polynomial of order $4K$ in z whose coefficients exhibit conjugate symmetry. Here, asymptotically, when L tends to infinity and the Fourier series truncation index, K , tends to infinity, the roots of $\bar{Q}_{PR}(z, \psi_2)$ for the N true values ψ_2 occur on the unit circle at $z = e^{j\psi_1}$ for the N true values of ψ_1 . Also, note that since the polynomial coefficients are conjugate symmetric, the roots occur in reciprocal pairs. If the roots are denoted as $\{\alpha_k(\psi_2)\}_{k=1}^{4K}$, then for each $\alpha_k(\psi_2)$ there will be another “redundant” root at $1/\alpha_k(\psi_2)$.

In practice, a search grid for the angular spreading parameter, ψ_2 is defined. Then, for each value on the grid, the search in mean angle, ψ_1 , is replaced by a polynomial rooting operation, where the angles of the “non-redundant” roots closest to the unit circle are chosen:

$$\hat{\zeta}_{PR} = \arg N \text{ lowest } \min_{\zeta} Q_{PR}(\psi_2) \quad (14)$$

$$Q_{PR}(\psi_2) = \|[1 - |\alpha_k(\psi_2)|]\|$$

The mean angle estimates are the angles associated with the roots satisfying the above criterion.

5. PERFORMANCE RESULTS

Results from computer simulations of the ML estimator (7), the ordinary MUSIC estimator (9), and the polynomial approach of (14) are now presented and compared against the Cramer-Rao Bound (CRB) as computed in [3]. Consider the case of an M element uniform linear array (ULA) with inter-element spacing d (relative to the signal wavelength) and coherently distributed scattered sources with uniform spatial density:

$$p(\theta|\psi) = \begin{cases} \frac{1}{2\psi_2} & |\theta - \psi_1| \leq \psi_2 \\ 0 & \text{otherwise} \end{cases}$$

For small ψ_2 , it is known that the elements of the spatial signature vector can be approximated as [1]:

$$b_m = e^{j2\pi d(m - [M+1]/2) \sin \psi_1} \cdot \text{sinc}(2d(m - [M+1]/2)\psi_2 \cos \psi_1)$$

where $\text{sinc}(\cdot) = \sin(\pi \cdot)/(\pi \cdot)$.

In particular, consider an $M = 4$ element ULA with half-wavelength inter-element spacing (i.e., $d = 0.5$) and a single coherently distributed scattered source with uniform spatial density. Computer simulations of the estimators described in Sections 3 and 4 based on 1000 independent realizations were carried out for various SNR's, mean angles, and polynomial orders. Performance results are shown in Figs. 1 and 2 for mean angle and spreading, respectively, as a function of SNR. The true parameters are: $\psi_1 = 0$

and $\psi_2 = \pi/10$. Results (bias, standard deviation, and root mean square error (RMSE)) are shown for the newly proposed technique for $K = 12$ and $K = 16$. It is seen that for sufficiently high SNR (and sufficiently high polynomial order in the case of the newly proposed technique) the CRB is achieved by all three estimators. Note that for ψ_1 the threshold SNR below which estimator performance deviates from the CRB is lower for the $K = 16$ polynomial rooting based estimator than the $K = 12$ estimator. Next, performance results are shown in Figs. 3 and 4 as a function of ψ_1 for fixed SNR = 10dB and $\psi_2 = \pi/10$. For mean angle values not near end-fire the relatively higher order $K = 16$ estimator achieves the CRB (as do the ML and MUSIC estimators). Closer to end-fire the $K = 12$ estimator is biased and deviates from the CRB.

6. CONCLUSIONS

This paper has presented a polynomial rooting approach to the localization of coherently distributed scattered sources. In particular, the search for the mean angle parameter in the MUSIC procedure is replaced by a polynomial rooting operation. Performance of the proposed technique was compared with that of the ML and ordinary MUSIC estimators as well as the CRB. It was seen that for sufficiently high SNR and polynomial order the performance of the newly proposed procedure often achieved the CRB. Future work will include development of an analogous polynomial rooting approach for incoherently scattered sources.

7. REFERENCES

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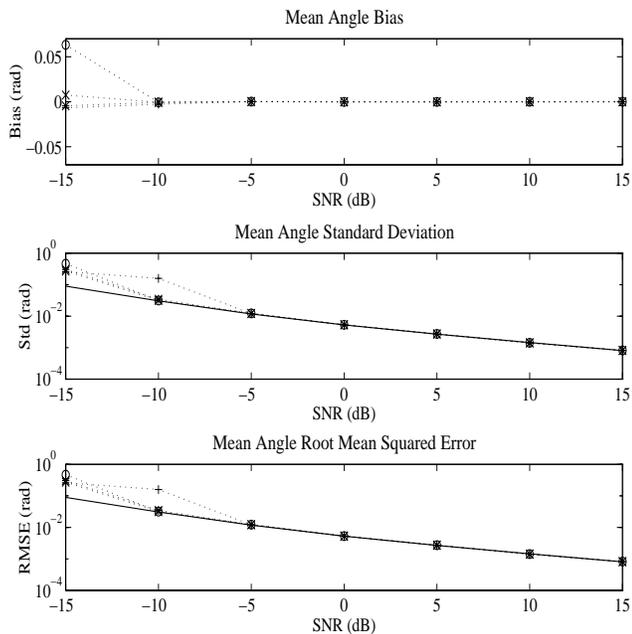


Figure 1: Estimator Performance for the mean angle parameter as a function of SNR. ($M = 4$, $\psi_1 = 0$, $\psi_2 = \pi/10$, $T = 1000$). (o) ML, (x) MUSIC, (+) $K = 12$ Polynomial Rooting, (*) $K = 16$ Polynomial Rooting, (solid) CRB.

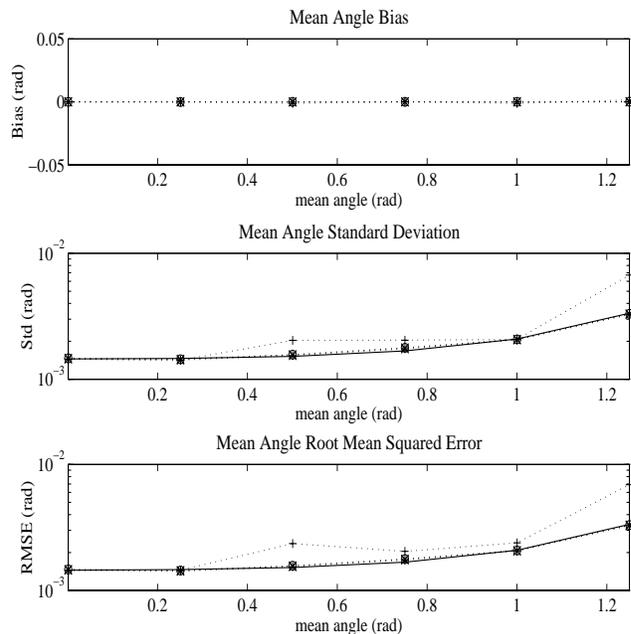


Figure 3: Estimator Performance for the mean angle parameter as a function of SNR. ($M = 4$, $\psi_1 = 0$, $\psi_2 = \pi/10$, $T = 1000$). (o) ML, (x) MUSIC, (+) $K = 12$ Polynomial Rooting, (*) $K = 16$ Polynomial Rooting, (solid) CRB.

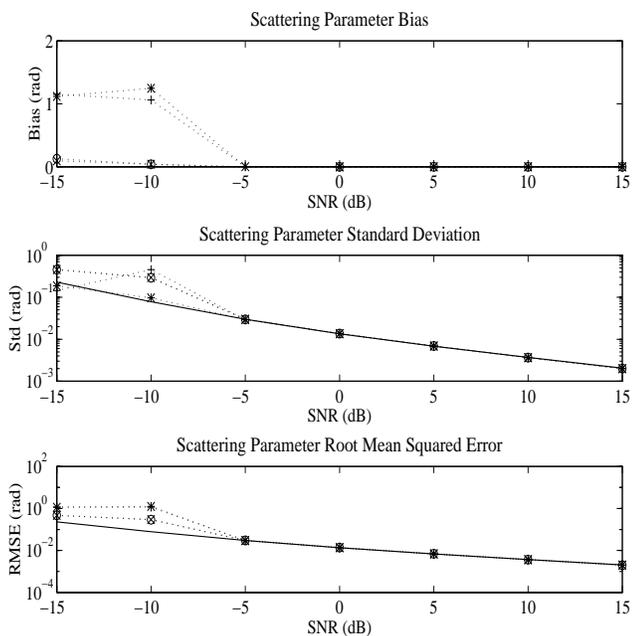


Figure 2: Estimator Performance for the angular spreading parameter as a function of SNR. ($M = 4$, $\psi_1 = 0$, $\psi_2 = \pi/10$, $T = 1000$). (o) ML, (x) MUSIC, (+) $K = 12$ Polynomial Rooting, (*) $K = 16$ Polynomial Rooting, (solid) CRB.

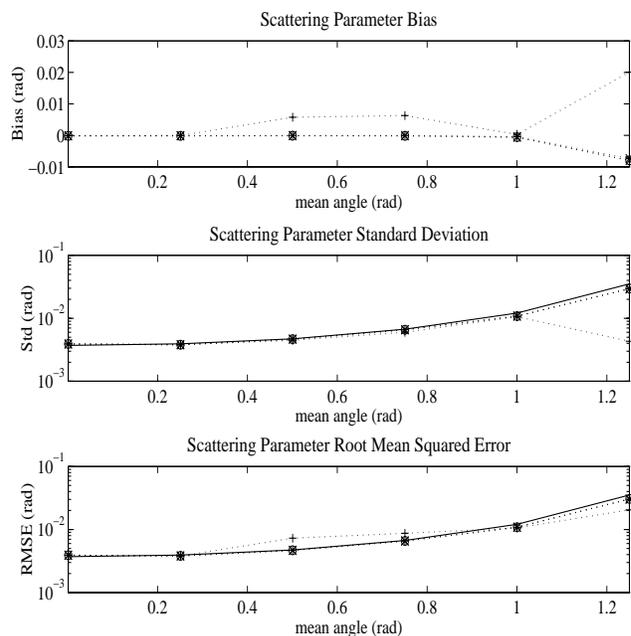


Figure 4: Estimator Performance for the angular spreading parameter as a function of SNR. ($M = 4$, SNR = 10dB, $\psi_2 = \pi/10$, $T = 1000$). (o) ML, (x) MUSIC, (+) $K = 12$ Polynomial Rooting, (*) $K = 16$ Polynomial Rooting, (solid) CRB.