# CONTINUOUS-TIME ENVELOPE CONSTRAINED FILTER DESIGN WITH INPUT UNCERTAINTY

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## Abstract

In an envelope-constrained filtering problem with uncertain input the set of possible inputs and the set of permissible outputs are each defined by envelopes or masks. This paper considers a continuous-time filter which in structure is comprised of an A/D converter, an FIR filter, a D/A converter and an analog post-filter. The object is to design the digital component of the filter structure so as to minimize the noise enhancement whilst satisfying the constraint that every signal in the input envelope evokes a response which stays in the output envelope.

## 1. INTRODUCTION

The continuous-time envelope-constrained (EC) filtering problem studied in [1-4], considers the design of a filter which minimizes noise enhancement subject to the constraint that the noiseless response $\psi_x$ of the filter to a specified excitation $x$ fits into a prescribed envelope. Filter design subject to envelope constraints has many applications, for example communications [5], radar/sonar detection, robust antenna and seismology [6].

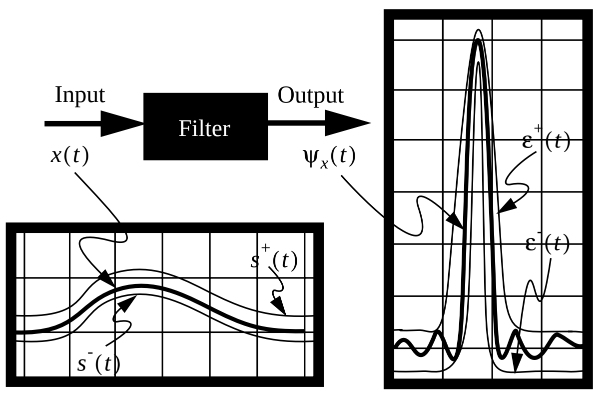


**Figure 1. EC filtering with uncertain input.**

Optimal EC filters, however, invariably lie on the boundary of the feasible set and consequently any disturbance on the input would violate the envelope constraints. To enforce robustness to input disturbances, we consider the more general case where the input $x$ is not specified exactly, but is known to lie within an input envelope described by the upper and lower boundaries $s^+$ and $s^-$ as shown in Figure 1 and seek a filter which forces all signals in the input mask to stay within $\varepsilon^-$ and $\varepsilon^+$. This problem is known as envelope-constrained with uncertain input (ECUI).

Early work in ECUI filtering [7-8] only considers an approximation using discrete-time signals and FIR filters. When the discrete-time outputs of these filters are converted to continuous-time, it is unlikely that these waveforms would still fit in the output envelope. Furthermore, the noise gains of these discrete-time filters are no longer optimal due to this conversion. In this paper, we formulate the (continuous-time) ECUI filtering problem for the hybrid filter shown in Figure 2, and present, for the first time, a novel technique to solve this problem.

## 2. PROBLEM FORMULATION

Consider the continuous-time filter realized using digital techniques as shown in Figure 2. This filter structure was also used in [4] for continuous-time EC filtering.

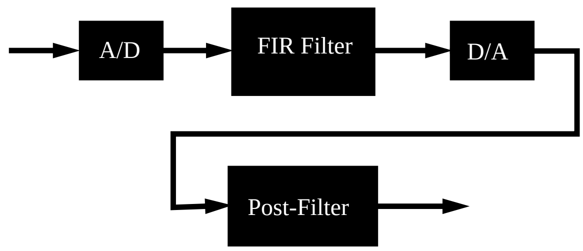


**Figure 2: Configuration of digital realization**

In what follows, it is assumed that the incoming signal is sampled at or above the Nyquist rate. To simplify matters, the quantization errors inherent in the A/D process are neglected. This assumption ensures the linearity of the system, since quantization is a non-linear operation.

### 2.1 Filter Output

Let $\boldsymbol{u} = [u_0, \ldots, u_{n-1}]^T \in \boldsymbol{R}^n$ be vector of coefficients of the FIR filter and $h(t)$ be the impulse response of the post-filter. It has been shown in [4] that the response of the hybrid filter to a signal $x$ is given by

$$\psi_x(t) = \sum_{i=-\infty}^{\infty} \sum_{j \in \hat{\Omega}_u} x[(i-j)\tau]u_j \Lambda(t-i\tau),\ t \in [0, \infty) \quad (1)$$

where $\Lambda(t) = \int_0^\infty \Pi(\lambda)h(t-\lambda)d\lambda$ is the response of the post-filter to the pulse $\Pi$, a rectangular pulse of length $\tau$, defined by

$$\Pi(t) = \begin{cases} 1,\ t \in [0, \tau] \\ 0,\ \text{otherwise} \end{cases}.$$

In practice, $\Pi$ has a shape that closely approximates the ideal rectangular pulse. Since we are only dealing with excitation with support $[0, \infty)$, the index $i$ in (1) can be taken from zero to infinity.

Assuming appropriate post-filtering such that (1) converges for all $t \in [0, \infty)$ (e.g. Bounded Input Bounded Output stability). Then, $\psi_x$ is bounded and continuous on $[0, \infty)$. Suppose that the sampled input (finite support) is given by the sequence $\{x(k\tau)\}_{k=0}^{m-1}$. Then, (see [4])

$$\psi_x(t) = \sum_{i=0}^{N-1} \sum_{j=0}^{n-1} x[(i-j)\tau]u_j \Lambda(t-i\tau) = \boldsymbol{y}_x^T(t)\boldsymbol{u} \quad (2)$$

where $N = n+m-1$, $\boldsymbol{y}_x^T(t) = [\Lambda(t), \ldots, \Lambda(t-(N-1)\tau)]X$,

$$X = \begin{bmatrix} x(0) & 0 & \ldots & 0 \\ \vdots & x(0) & & \vdots \\ x((m-1)\tau) & \vdots & \ddots & 0 \\ 0 & x((m-1)\tau) & & x(0) \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & x((m-1)\tau) \end{bmatrix}_{N \times n}$$

### 2.2 Output Noise Power

A simple cost function we could use for the optimization is the Euclidean norm of the discrete-time filter. However, unlike the discrete-time case [7], the output noise power of this system is not directly proportional to this norm. Assuming stationary input noise samples, it has been shown in [4] that the output noise

$$\xi(t) = \sum_{l=-\infty}^{\infty} \sum_{j=0}^{n-1} \Lambda(t-l\tau)n[(l-j)\tau]u_j$$

is cyclostationary with period $\tau$ and the average output noise power is directly proportional to

$$\|\boldsymbol{u}\|_L^2 \equiv \boldsymbol{u}^T L \boldsymbol{u} \quad (3)$$

where $L$ is a positive definite matrix defined by

$$L = \begin{bmatrix} L_0 & L_1 & \ldots & L_{n-2} & L_{n-1} \\ L_1 & L_0 & & & L_{n-2} \\ \vdots & & \ddots & & \vdots \\ L_{n-2} & & & L_0 & L_1 \\ L_{n-1} & L_{n-2} & \ldots & L_1 & L_0 \end{bmatrix},$$

$$L_j = \frac{N_0}{\tau}\int_{-\infty}^{\infty} \Lambda(\varsigma)\Lambda(\varsigma - j\tau)d\varsigma,\ j = 0, \ldots n-1.$$

### 2.3 Problem statement

Let $S(s, \theta)$ denote the set of possible input signals, i.e.

$$S(s, \theta) = \{x : |x(\tau) - s(\tau)| \le \theta(\tau), \forall \tau \in \Omega_s\}$$

where $s \equiv 0.5(s^+ + s^-)$, $\theta \equiv 0.5(s^+ - s^-)$ and $\Omega_s$ is the support of the input signals. Then we seek a minimum $L$-weighted-norm vector of filter coefficients such that the filter output $\psi_x$ evoked by every signal $x$ in $S(s, \theta)$ satisfies

$$\varepsilon^-(t) \le \psi_x(t) \le \varepsilon^+(t)$$

for all $t$ on an interval $\Omega_c$. Thus, the ECUI filtering problem can be written as

$$\begin{aligned} &\min \|\boldsymbol{u}\|_L^2,\ \boldsymbol{u} \in \boldsymbol{R}^n \\ &\text{subject to } \left|\boldsymbol{y}_x^T(t)\boldsymbol{u} - d(t)\right| \le \varepsilon(t), \forall t \in \Omega_c, \forall x \in S(s, \theta) \end{aligned} \quad \text{(P.1-a)}$$

where $d \equiv 0.5(\varepsilon^+ + \varepsilon^-)$, $\varepsilon \equiv 0.5(\varepsilon^+ - \varepsilon^-)$.

## 3. SOLUTION METHOD

The description of the feasible region in terms of $S(s, \theta)$ is adequate for characterizing the problem, but not particularly useful for computational purposes. To evaluate the feasibility of a filter, one would need to compute its response to every signal $x$ in the possible input set $S(s, \theta)$. This is intractable and there is, apparently, no standard numerical techniques for handling problems of this type. In this section we present a novel technique for solving P.1-a by transforming it into a QP problem with affine functional inequality constraints.

To solve the ECUI problem, it would be necessary to have an equivalent but more explicit expression for the constraints of P.1-a which does not involve $x$. Since the set of possible input is completely characterized by $s$ and $\theta$, we want an expression which would involve only $d$, $\varepsilon$, $s$ and $\theta$. This is provided by the following result (the proof can be found in [9]).

***Theorem 3.1.*** $\left|\boldsymbol{y}_x^T(t)\boldsymbol{u} - d(t)\right| \le \varepsilon(t), \forall t \in \Omega_c, \forall x \in S(s, \theta)$ *if and only if* $\left|\boldsymbol{y}_s^T(t)\boldsymbol{u} - d(t)\right| + \boldsymbol{z}^T(t)|\boldsymbol{u}| \le \varepsilon(t), \forall t \in \Omega_c$, *where*

$$\boldsymbol{z}^T(t) = [|\Lambda(t)|, \ldots, |\Lambda(t-(n+m-1)\tau)|]\Theta,$$

$$\Theta = \begin{bmatrix} \theta(0) & 0 & \dots & 0 \\ \vdots & \theta(0) & & \vdots \\ \theta((m-1)\tau) & \vdots & \ddots & 0 \\ 0 & \theta((m-1)\tau) & & \theta(0) \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \theta((m-1)\tau) \end{bmatrix}_{N \times n}$$

Thus P.1-a can be equivalently posed as

$$\min \boldsymbol{u}^T L \boldsymbol{u} = \|\boldsymbol{u}\|_L^2, \boldsymbol{u} \in \boldsymbol{R}^n$$
$$\text{subject to } \left|\boldsymbol{y}_s^T(t)\boldsymbol{u} - d(t)\right| + \boldsymbol{z}^T(t)|\boldsymbol{u}| \le \varepsilon(t), \forall t \in \Omega_c \qquad \text{(P.1-b)}$$

This is a finite-dimensional QP problem with functional inequality constraints. It can be easily verified that the constraint set for this problem is convex. Hence, P.1-b has a unique solution since the cost function is strictly convex. Note that P.1-b is a non-smooth problem and is difficult to solve because of the non-differentiable term $|\boldsymbol{u}|$ . However, it can be converted to the following smooth problem with twice as many dimensions.

$$\min f(\boldsymbol{v}) = \boldsymbol{v}^T H \boldsymbol{v}, \boldsymbol{v} \in \boldsymbol{R}^{2n} \qquad \text{(P.2)}$$

subject to

$$\boldsymbol{y}_s^T(t)[I_n, -I_n]\boldsymbol{v} + \boldsymbol{z}^T(t)[I_n, I_n]\boldsymbol{v} - \varepsilon^+(t) \le 0, \forall t \in \Omega_c$$
$$-\boldsymbol{y}_s^T(t)[I_n, -I_n]\boldsymbol{v} + \boldsymbol{z}^T(t)[I_n, I_n]\boldsymbol{v} + \varepsilon^-(t) \le 0, \forall t \in \Omega_c$$
$$-\boldsymbol{v} \le 0$$

where

$$H = \begin{bmatrix} L & \rho I_n - L \\ \rho I_n - L & L \end{bmatrix}_{2n \times 2n},$$

$I_n$ is an $n \times n$ identity matrix and $0 < \rho < 2\lambda_{min}(L)$ .

Note that $H$ is positive definite. To see this we call upon Theorem 7.7.6 of [10], which asserts that for $H$ to be positive definite, it is necessary and sufficient that

$$L - (\rho I_n - L)L^{-1}(\rho I_n - L) = \rho^2[(2/\rho)I_n - L^{-1}] \qquad (4)$$

be positive definite. Let $\lambda_i$ be the eigen values of $L$ . Then the eigen values of (4) are $2/\rho - 1/\lambda_i > 0$ , $i = 1, \dots, n$ . Since $0 < \rho < 2\lambda_{min}(L)$ , it is follows that (4) is positive definite.

Problem P.2 is also a QP problem but with affine functional inequality constraints. Moreover, since the cost function is strictly convex (because $H$ is positive definite), this problem has a unique solution (if one exists).

***Theorem 3.2.****If* $\boldsymbol{u}^*$ *and* $\boldsymbol{v}^*$ *are the optimal solution to problems P.1 and P.2 respectively then* $\boldsymbol{u}^* = [I_n, -I_n]\boldsymbol{v}^*$ .

This result establishes the equivalence of P.1 and P.2 by stating the relationship between their respective solutions $\boldsymbol{u}^*$ and $\boldsymbol{v}^*$ (see Appendix for proof). Problem P.2 can easily be solved by approximating the continuum of constraints by a finite number of constraints as follows

$$\boldsymbol{y}^T(t_i)[I_n, -I_n]\boldsymbol{v} + \boldsymbol{z}^T(t_i)[I_n, I_n]\boldsymbol{v} - \varepsilon^+(t_i) \le 0, \forall i = 0, \dots, M-1$$
$$-\boldsymbol{y}^T(t_i)[I_n, -I_n]\boldsymbol{v} + \boldsymbol{z}^T(t_i)[I_n, I_n]\boldsymbol{v} + \varepsilon^-(t_i) \le 0, \forall i = 0, \dots, M-1$$
$$-\boldsymbol{v} \le 0$$

Hence, it can be guaranteed that the envelope constraints are satisfied at the instances $t_i, i = 0, \dots, M-1$ (if there exists a feasible solution). This approximate problem is a standard QP problem with a positive definite cost and can be efficiently solved by well established algorithms such as QP via active set strategy.

## 4. EXAMPLE

Consider the compression of a 13 bit barker coded signal with an allowable error of 3% as shown in Figure 3 (a unit on the time axis corresponds to one bit interval $\beta$ .) We require all responses to signals within this input mask to fit in an output mask with allowable sidelobes of $0.025$ and a mainlobe peak of $0.69 \pm 0.075$ as shown in Figure 4.

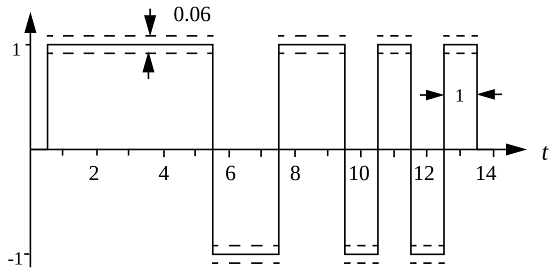


**Figure 3. 13 -bit Barker-coded signal with input mask**

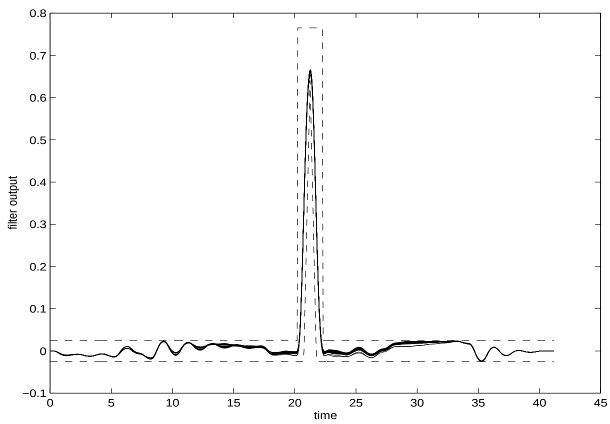


**Figure 4. Output mask and outputs of optimal filter**

For this example we use a 27-tap FIR filter and a Bessel post-filter of 6th order with cut-off frequency $\omega_c = 2\pi/\beta$ . To obtain the approximate solution, we have constrained the output at every $t_i = i\beta/8$ . The noise gain of optimal

ECUI filter is $\|\boldsymbol{u}^*\|_L^2 = 0.02654$. Figure 4 also shows the filter's responses to signals which were randomly perturbed about the nominal input but still fit inside the input mask. Observe that these responses stay within the boundary of the output envelope.

## 5. CONCLUSIONS

Proposed in this note is a continuous-time filter which consists of an A/D converter, an FIR filter, a D/A converter and an analog post-filter for envelope constrained filtering with input uncertainty. It has been shown that the original ECUI problem can be converted into a QP problem with affine functional inequality constraints. This novel transformation thus allows the seemingly intractable ECUI problem to be solved in a straightforward manner using well established numerical routines such as QP via active set strategy. The technique has been successfully applied to a radar pulse compression example as demonstrated by the numerical study.

## 6. APPENDIX

***Lemma***. *Let* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ *be the mappings defined by*

$$[\boldsymbol{p}(\boldsymbol{u})]_k = \max(\boldsymbol{u}_k, 0)$$
$$[\boldsymbol{q}(\boldsymbol{u})]_k = \max(-\boldsymbol{u}_k, 0).$$

*Then,*

- *(i)* $\boldsymbol{p}(\boldsymbol{u}) \geq 0, \boldsymbol{q}(\boldsymbol{u}) \geq 0$.
- *(ii)* $[\boldsymbol{p}(\boldsymbol{u})]^T \boldsymbol{q}(\boldsymbol{u}) = 0$.
- *(iii)* $\boldsymbol{p}(\boldsymbol{u}) - \boldsymbol{q}(\boldsymbol{u}) = \boldsymbol{u}$.
- *(iv)* $\boldsymbol{p}(\boldsymbol{u}) + \boldsymbol{q}(\boldsymbol{u}) = |\boldsymbol{u}|$.
- *(v)* *for each* $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{R}^n$ *with* $\boldsymbol{u}, \boldsymbol{v} \geq 0$ *there exists a unique non-negative function* $\boldsymbol{r}: \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ *such that* $|\boldsymbol{u} - \boldsymbol{v}| = \boldsymbol{u} + \boldsymbol{v} - \boldsymbol{r}(\boldsymbol{u}, \boldsymbol{v})$

Parts (i)-(iv) is straight forward. For part (v), it suffices to define

$$[\boldsymbol{r}(\boldsymbol{u}, \boldsymbol{v})]_k = \boldsymbol{u}_k(1 - \operatorname{sgn}(\boldsymbol{u}_k - \boldsymbol{v}_k)) + v(t)(1 + \operatorname{sgn}(\boldsymbol{u}_k - \boldsymbol{v}_k)).$$

***Proof of Theorem 3.2***.

Suppose $\boldsymbol{v}^*$ is the optimal solution of P.2. Let $\boldsymbol{x}^* = [I_n, 0_n]\boldsymbol{v}^*$ and $\boldsymbol{y}^* = [0_n, I_n]\boldsymbol{v}^*$ so that $\boldsymbol{v}^* = [\boldsymbol{x}^{*T}, \boldsymbol{y}^{*T}]^T$ and $\boldsymbol{x}^* - \boldsymbol{y}^* = [I_n, -I_n]\boldsymbol{v}^*$. Then

$$f(\boldsymbol{v}^*) = \|\boldsymbol{x}^* - \boldsymbol{y}^*\|_L^2 + 2\rho \boldsymbol{x}^{*T}\boldsymbol{y}^* \geq \|\boldsymbol{x}^* - \boldsymbol{y}^*\|_L^2. \qquad (5)$$

Let $\boldsymbol{g} = \boldsymbol{p}(\boldsymbol{x}^* - \boldsymbol{y}^*)$, $\boldsymbol{h} = \boldsymbol{q}(\boldsymbol{x}^* - \boldsymbol{y}^*)$. By (ii) and (iii)

$$f([\boldsymbol{g}^T, \boldsymbol{h}^T]^T) = \|\boldsymbol{g} - \boldsymbol{h}\|_L^2 + 2\rho \boldsymbol{g}^T\boldsymbol{h} = \|\boldsymbol{g} - \boldsymbol{h}\|_L^2 = \|\boldsymbol{x}^* - \boldsymbol{y}^*\|_L^2$$

Moreover, from (i) $\boldsymbol{g}, \boldsymbol{h} \geq 0$ and by (iv)-(v) we have

$$\left|\boldsymbol{y}_s^T(t)(\boldsymbol{g} - \boldsymbol{h}) - d(t)\right| + \boldsymbol{z}^T(t)(\boldsymbol{g} + \boldsymbol{h})$$
$$= \left|\boldsymbol{y}_s^T(t)(\boldsymbol{x}^* - \boldsymbol{y}^*) - d(t)\right| + \boldsymbol{z}^T(t)(\boldsymbol{x}^* + \boldsymbol{y}^* - \boldsymbol{r}(\boldsymbol{x}^*, \boldsymbol{y}^*))$$
$$\leq \left|\boldsymbol{y}_s^T(t)[I_n, -I_n]\boldsymbol{v}^* - d\right| + \boldsymbol{z}^T(t)[I_n, I_n]\boldsymbol{v}^* \leq \varepsilon(t)$$

that is $[\boldsymbol{g}^T, \boldsymbol{h}^T]^T$ satisfies the constraints of P.2. Hence by the optimality of $\boldsymbol{v}^*$,

$$f(\boldsymbol{v}^*) \leq f([\boldsymbol{g}^T, \boldsymbol{h}^T]^T) = \|\boldsymbol{x}^* - \boldsymbol{y}^*\|_L^2$$

and thus using (5) yields $f(\boldsymbol{v}^*) = \|\boldsymbol{x}^* - \boldsymbol{y}^*\|_L^2$.

Since $\boldsymbol{u}^* = \boldsymbol{x}^* - \boldsymbol{y}^*$, from (v) we have

$$\left|\boldsymbol{y}_s^T(t)\boldsymbol{u}^* - d(t)\right| + \boldsymbol{z}^T(t)|\boldsymbol{u}^*|$$
$$= \left|\boldsymbol{y}_s^T(t)(\boldsymbol{x}^* - \boldsymbol{y}^*) - d(t)\right| + \boldsymbol{z}^T(t)(\boldsymbol{x}^* + \boldsymbol{y}^* - \boldsymbol{r}(\boldsymbol{x}^*, \boldsymbol{y}^*)) \leq \varepsilon(t)$$

hence $\boldsymbol{u}^*$ satisfies the envelope constraints. Now for any $\boldsymbol{u}$ satisfying the envelope constraints, by (iii) and (iv)

$$\left|\boldsymbol{y}_s^T(t)(\boldsymbol{p}(\boldsymbol{u}) - \boldsymbol{q}(\boldsymbol{u})) - d(t)\right| + \boldsymbol{z}^T(t)(\boldsymbol{p}(\boldsymbol{u}) + \boldsymbol{q}(\boldsymbol{u}))$$
$$= \left|\boldsymbol{y}_s^T(t)\boldsymbol{u} - d(t)\right| + \boldsymbol{z}^T(t)|\boldsymbol{u}| \leq \varepsilon(t)$$

This together with (i) means $[\boldsymbol{p}(\boldsymbol{u})^T, \boldsymbol{q}(\boldsymbol{u})^T]^T$ satisfies the constraints of P.2. By the optimality of $\boldsymbol{v}^*$, (ii) and (iii) we arrive at

$$\|\boldsymbol{u}^*\|_L^2 = \|\boldsymbol{x}^* - \boldsymbol{y}^*\|_L^2 = f(\boldsymbol{v}^*) \leq f([\boldsymbol{p}(\boldsymbol{u})^T, \boldsymbol{q}(\boldsymbol{u})^T]^T) = \|\boldsymbol{u}\|_L^2$$

Thus $\boldsymbol{u}^* = [I_n, -I_n]\boldsymbol{v}^*$ is the optimal solution of P.1. ❐

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