Optimum Finite-Length LTI Transmit Filters for ISI-Channels

Naofal Al-Dhahir
GE Corporate R &D Center
Niskayuna, NY 12309
Email : aldhahir@birch.crd.ge.com

Abstract

Optimum FIR transmit filters for symbol–by–symbol transmission on linear dispersive additive–Gaussian–noise channels are derived by maximizing the channel throughput, subject to a fixed average input energy constraint. This maximized throughput is compared with that achievable with water–pour and flat transmit filters. The effect of transmit filter optimization on the receiver performance is investigated by considering the popular MMSE–DFE structure.

1 Introduction

Maximizing the achievable throughput of noisy intersymbol interference (ISI) channels requires optimization of both the transmitter and receiver ends of the communication system. In multicarrier transceivers, an orthogonal transformation (such as FFT) is used to convert the wideband ISI channel into a large number of parallel ISI–free narrow–band subchannels that can be individually decoded. On the other hand, single carrier transceivers commonly employ finite–impulse–response (FIR) filters at both the transmitter and receiver ends to mitigate the ISI and noise.

The FIR minimum–mean–square–error decision feedback equalizer (MMSE–DFE) is a widely–used receiver structure on severe–ISI channels. Optimizing the MMSE–DFE filter settings for the infinite–length and finite–length cases was treated in [1, 2] and [3], respectively. While there have been several studies on the transmitter optimization problem, they either assume an infinite–length transmit filter as in [1, 2] or block–by–block transmission as in [4]. For block–based transmission, the throughput–maximizing input covariance process was shown in [4] to be *nonstationary* and the eigenvalues of its auto–correlation matrix obey a water–pour distribution. Although this input covariance process was shown to have a special structure that results in a generalized constant–parameter lattice filter implementations of the transmit filter, the synthesis procedure is somewhat involved. Moreover, FIR modeling filters of this nonstationary process are *time–varying* and hence costly to implement.

Our objective in this paper is to maximize the throughput of noisy ISI linear channels by passing the white input sequence through an FIR linear time–invariant (LTI) transmit filter that introduces correlation between the samples of its input sequence while preserving its stationarity and average input energy.

2 Transmit Filter Optimization

2.1 Input-Output Model

We adopt the standard discrete–time representation of an additive–noise dispersive channel given by

$$\mathbf{y}_k = \sum_{m=0}^{\nu} \mathbf{h}_m x_{k-m} + \mathbf{n}_k \ , \tag{1}$$

where $\mathbf{h}_m \stackrel{def}{=} \left[\begin{array}{ccc} h_{l-1,m} & \cdots & h_{0,m} \end{array} \right]^t$ is the m^{th} (vector) channel coefficient, assuming an oversampling factor of l, ν is called the channel memory, and $(.)^t$ denotes the transpose. We assume a continuous transmission bandwidth and perfect knowledge of the channel and noise characteristics at the transmitter and receiver ends. Both the input sequence, $\{x_k\}$, and the noise sequence, $\{\mathbf{n}_k\}$, are assumed to be stationary, zero–mean, and have (non–singular) Toeplitz auto–correlation matrices denoted by $\mathbf{R}_{xx}$ and $\mathbf{R}_{nn}$, respectively. The input sequence is generated by an FIR transmit filter according to

$$x_k = \sum_{n=0}^{\nu_t} p_n \epsilon_{k-n} \ , \tag{2}$$

where $\{p_i\}_{i=0}^{\nu_t}$ are the transmit filter coefficients and $\{\epsilon_k\}$ is a white unit–energy sequence.

Over any block of N output symbols, (1) becomes

$$\mathbf{y}_{k+N-1:k} = \mathbf{H}\mathbf{x}_{k+N-1:k-\nu} + \mathbf{n}_{k+N-1:k} \ ,$$

where $\mathbf{H}$ is a fully–windowed Toeplitz channel matrix whose first block–row is equal to the $(\nu+1)$ channel impulse response (CIR) coefficients appended by zeros. Similarly, the vector representation of (2) is

$$\mathbf{x}_{k+N-1:k-\nu} = \mathbf{P}\epsilon_{k+N-1:k-\nu-\nu_t} \ , \tag{3}$$

2.2 Maximizing the Channel Throughput

It was shown in [4] that for Gaussian input and noise sequences, the (normalized per input symbol) channel throughput is given by

$$\bar{I} = \frac{1}{(N+\nu)}\log_2(|\mathbf{I}_{N+\nu} + \mathbf{H}^*\mathbf{R}_{nn}^{-1}\mathbf{H}\mathbf{R}_{xx}|) \ , \tag{4}$$

where $\mathbf{I}_{N+\nu}$ is the identity matrix of size $(N+\nu)$ and $(.)^*$ denotes the complex–conjugate transpose.

We want to find the optimum $\mathbf{R}_{xx}$ that maximizes (4) **subject** to the constraint of a unit–energy transmit filter, i.e., $\sum_{i=0}^{\nu_t}|p_i|^2 = 1$. This constraint holds the average input energy to the channel at its level with the white input sequence $\{\epsilon_k\}$.

Remark

From Equation (3), $\mathbf{R}_{xx} = \mathbf{P}\mathbf{P}^*$ which guarantees that the input correlation matrix will be a Hermitian positive semi–definite matrix of size $(N+\nu)$, which is *independent* of the transmit filter length. Perhaps less obvious is the fact that since $\mathbf{P}$ is a *fully-windowed* Toeplitz matrix, $\mathbf{R}_{xx}$ will also be Toeplitz with $R_{xx}(i,j) = \sum_{k=0}^{\nu_t} p_k p_{k+|j-i|}^*$. In addition, the constraint $\sum_{i=0}^{\nu_t}|p_i|^2 = 1$ is equivalent to $\frac{1}{N+\nu}trace(\mathbf{R}_{xx}) = 1$. Hence, the FIR transmit filter preserves the stationarity and average input energy level of its input sequence $\{\epsilon_k\}$.

3 Application to MMSE-DFE

3.1 Decision-Point SNR

As shown in Figure 1, the FIR MMSE–DFE consists of a fractionally–spaced feedforward filter that has (lN) taps and is denoted by the vector $\mathbf{w}^*$, and a symbol–spaced feedback filter that is *assumed to have* N_b *strictly causal taps* denoted by $\{-b_1, -b_2, \cdots, -b_{N_b}\}$. For analytical convenience, we define the augmented vector $\tilde{\mathbf{b}}^* \stackrel{def}{=} \left[\ \mathbf{0}_{1\times\Delta} \quad 1 \quad b_1^* \quad \cdots \quad b_{N_b}^*\ \right]$, where Δ is the delay of feedforward filter.

We showed in detail in [3] that the optimum feedback and feedforward filter settings are given by

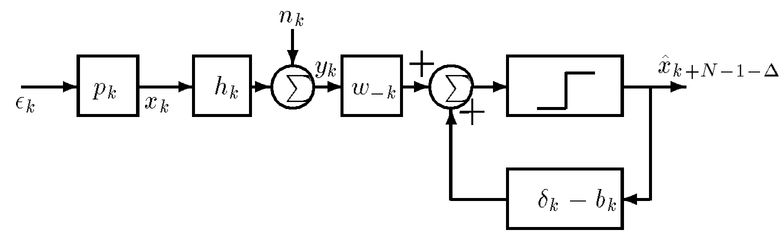

Figure 1: Block Diagram of the MMSE-DFE

$$\tilde{\mathbf{b}}_{opt.} = \mathbf{L}\mathbf{e}_{(\Delta_{opt.}+1)} \tag{5}$$
$$\mathbf{w}_{opt.}^* = d_{\Delta_{opt.}}^{-1}\mathbf{e}_{(\Delta_{opt.}+1)}^*\mathbf{L}^{-1}\mathbf{H}^*\mathbf{R}_{nn}^{-1} \ , \tag{6}$$

where $\mathbf{e}_i$ denotes the i^{th} unit column vector. The optimum delay, $\Delta_{opt.}$, is given by

$$\Delta_{opt.} = argmax_{0\le i\le(N+\nu-1)}\{d_i\} \ . \tag{7}$$

The matrix $\mathbf{L}$ that appears in (5) and (6) and the scalars $\{d_i\}$ in (6) and (7) are defined by the following *Cholesky factorization* [3]

$$\mathbf{R} \stackrel{def}{=} \mathbf{R}_{xx}^{-1} + \mathbf{H}^*\mathbf{R}_{nn}^{-1}\mathbf{H} \stackrel{def}{=} \mathbf{L}\mathbf{D}\mathbf{L}^* \ , \tag{8}$$

where $\mathbf{L}$ is a lower–triangular monic (has ones along the main diagonal) matrix and $\mathbf{D}$ is a diagonal matrix with entries $d_0, d_1, \cdots, d_{N+\nu-1}$. Furthermore, the *unbiased* decision–point SNR of the FIR MMSE–DFE is given by [3]

$$SNR_{MMSE-DFE,U} = d_{\Delta_{opt.}} - 1 \ . \tag{9}$$

Remarks on Computational Complexity

- The FIR transmit filter optimization problem of Section 2.2 is a constrained and nonlinear problem that we solved numerically using the *sequential quadratic programming* (SQP) algorithm. This algorithm uses an approximation of the Hessian of (4), computed using a quasi–Newton updating procedure to solve a quadratic programming subproblem and establish a search direction for a line search procedure.

- It was shown in [3] that the matrix $\mathbf{R}$ defined in (8) is *structured* and that its *displacement rank* [5] is upper–bounded by

$$\rho(\mathbf{R}) \le \rho(\mathbf{R}_{xx}^{-1}) + \rho(\mathbf{H}^*\mathbf{R}_{nn}^{-1}\mathbf{H}) = 4 \ ,$$

for stationary input and noise sequences. Hence, the Cholesky factorization needed to compute the optimal MMSE–DFE settings can be computed in $O(4(N+\nu)^2)$ instead of $O((N+\nu)^3)$ operations.

3.2 A Lower-Bound on $SNR_{MMSE-DFE}$

A lower–bound on $SNR_{MMSE-DFE}$ can be established as follows. Starting with Equation (4)

$$\begin{aligned}
\bar{I} &= \frac{1}{(N+\nu)}\log_2(|\mathbf{R}_{xx}^{-1}+\mathbf{H}^*\mathbf{R}_{nn}^{-1}\mathbf{H}||\mathbf{R}_{xx}|) \\
&= \frac{1}{(N+\nu)}\log_2(|\mathbf{R}_{xx}|\prod_{i=0}^{N+\nu-1} d_i) \;:\; \text{Using}(8) \\
&\leq \frac{1}{(N+\nu)}\log_2(|\mathbf{R}_{xx}|(d_{\Delta_{opt}})^{N+\nu}) \;:\; \text{Using}(7) \\
&= \log_2(|\mathbf{R}_{xx}|^{\frac{1}{(N+\nu)}}) + \log_2(d_{\Delta_{opt}}) \\
&\leq \log_2(\frac{1}{(N+\nu)}trace(\mathbf{R}_{xx})) + \log_2(d_{\Delta_{opt}}) \\
&= \log_2(SNR_{MMSE-DFE,U}+1) \;:\; \text{Using}(9)\,,
\end{aligned}$$

with equality achieved as N becomes infinite. Therefore, by maximizing $\bar{I}$ we're maximizing the minimum value that $SNR_{MMSE-DFE}$ can ever attain under the given channel and noise conditions.

4 Simulation Results

In our computer simulations we consider the CIR $h(D) = 0.407+0.815D+0.407D^2$, also used in [6], and will be denoted here by Channel 1. The transmit filter memory is assumed equal to the channel memory, i.e., $\nu_t = 2$. Figures 2 and 3 show that the optimized 3–tap FIR filter achieves a throughput improvement of about 0.2 bits/symbol over the flat transmit filter case. The maximum throughput achieved by the theoretically–optimum *water–pour* transmit filter [7, 4] is also shown as an upper–bound. Figure 4 shows that the channel throughput achieved with a 15–tap FIR transmit filter is within 0.05 bits/symbol from this upper–bound!

Next, we investigated the effect of transmit filter optimization on the unbiased decision–point SNR of the MMSE–DFE defined in (9). Figure 5 shows that appreciable performance improvements are achieved, especially at low input SNR. At very high input SNR, transmit filter shaping becomes less important and even the water–pour energy distribution becomes essentially flat [2, 4]. On the other hand, the effect of transmit filter optimization becomes more significant for highly–dispersive channels, as shown in Figure 6 for the EPR6 channel $h(D) = \frac{(1+D)^4(1-D)}{\sqrt{28}}$, which we call Channel 2.

Finally, we plot the power spectrum of Channel 1 and that of its optimized 3–tap FIR transmit filter for $N = 10$ and an input SNR of 20 dB which was calculated to be $p(D) = 0.7963 + 0.5544D - 0.2419D^2$. As expected, the optimized transmit filter has high gain in

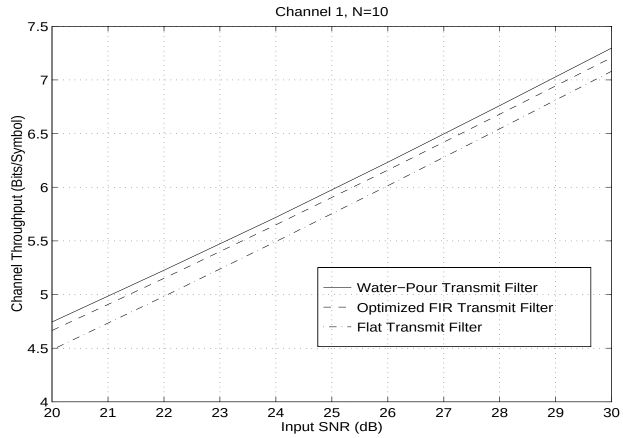

Figure 2: Channel throughput versus input SNR with water-pour, flat, and optimized FIR transmit filters

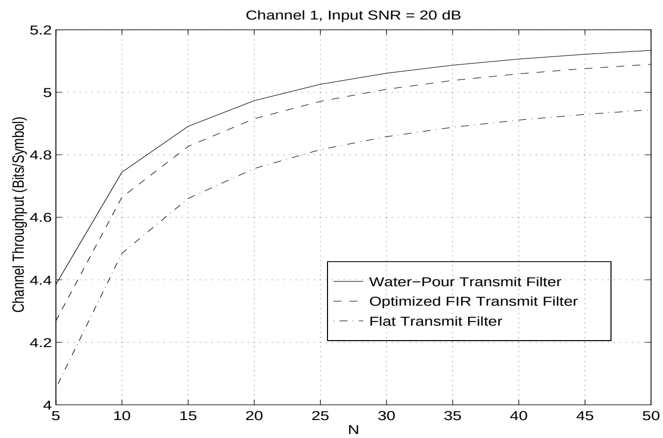

Figure 3: Channel throughput versus N with water-pour, flat, and optimized FIR transmit filters

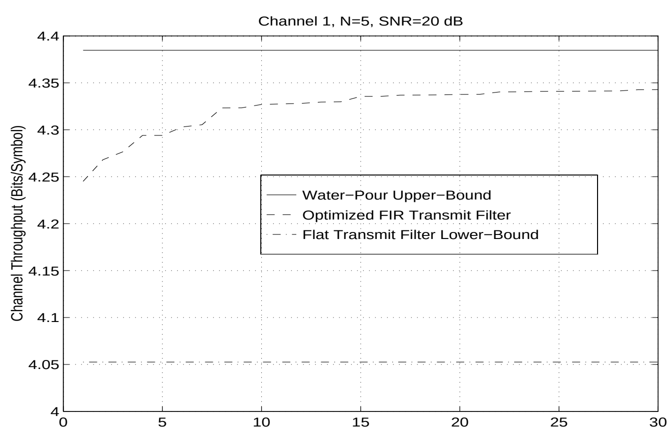

Figure 4: Channel throughput versus ν_t with water-pour, flat, and optimized FIR transmit filters

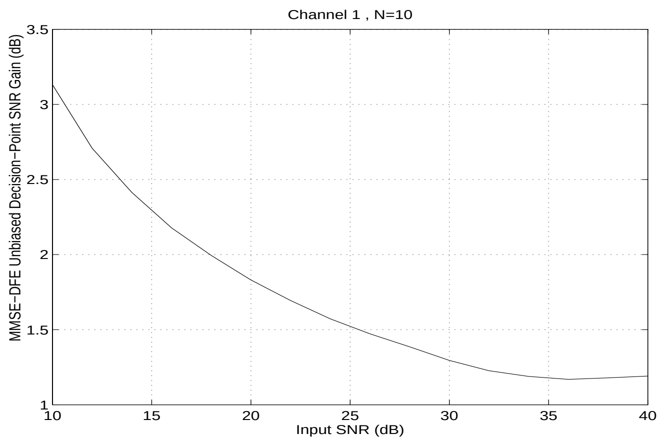

Figure 5: $SNR_{MMSE-DFE,U}$ improvement with transmit filter optimization for Channel 1

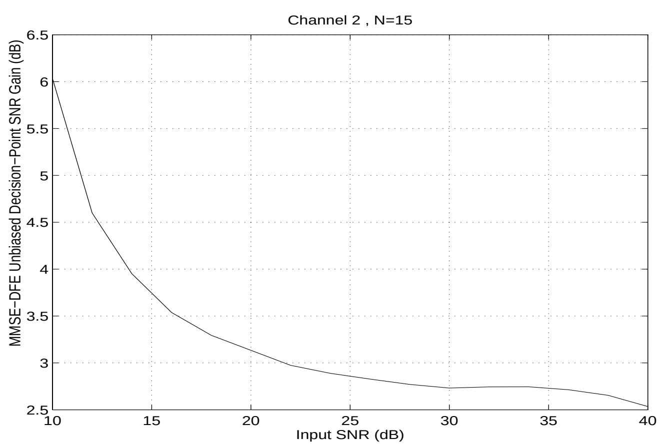

Figure 6: $SNR_{MMSE-DFE,U}$ improvement with transmit filter optimization for Channel 2

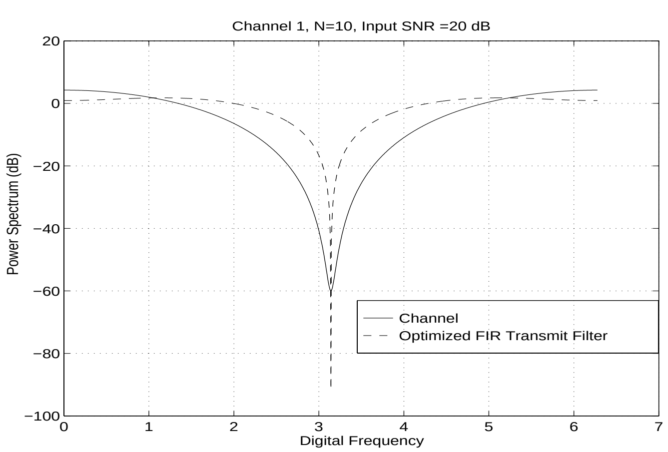

Figure 7: Power spectrum of Channel 1 and its optimized 3–tap FIR transmit filter

the "good" portions of the channel and low gain in the "bad" portions to maximize throughput.

5 Conclusions

We showed that substantial improvements in the decision–point SNR of the MMSE–DFE can be achieved by optimizing the transmit filter. The optimum FIR transmit filters derived in this paper maximize the channel throughput while preserving the stationarity and average input energy level of the input sequence. Because of the Toeplitz structure of the optimized input correlation matrix, alternative all–pole or lattice implementations of the transmit filter can be computed efficiently using the Levinson algorithm [8].

References

[1] J. Salz. "Optimum Mean-Square Decision Feedback Equalization". *Bell System Tech. J.*, 52(8), October 1973.

[2] J.M.Cioffi and G. Dudevoir and M.V. Eyuboglu and G.D. Forney, Jr. "Minimum Mean-Square-Error Decision Feedback Equalization and Coding - Parts I and II". *IEEE Transactions on Communications*, October 1995.

[3] N. Al-Dhahir and J.M.Cioffi. "MMSE Decision Feedback Equalizers : Finite-Length Results". *IEEE Transactions on Information Theory*, 41(4):961–976, July 1995.

[4] N. Al-Dhahir and J.M.Cioffi. "Block Transmission over Dispersive Channels : Transmit Filter Optimization and Realization, and MMSE-DFE Receiver Performance". *IEEE Trans. on Information Theory*, pages 137–160, January 1996.

[5] T. Kailath and A.H.Sayed. "Displacement Structure: Theory and Applications". *SIAM Review*, 37(3):297–386, September 1995.

[6] J. Proakis. *Digital Communications*. McGraw Hill, 1989.

[7] R. Gallager. *"Information Theory and Reliable Communication"*. Wiley, 1968.

[8] S. Haykin. *"Adaptive Filter Theory"*. Prentice Hall, 1991. 2nd Edition.