DETECTION AND ESTIMATION OF CHANGES IN THE PARAMETERS OF A CHIRP SIGNAL

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ABSTRACT

The problem is the detection and the estimation of abrupt changes in a chirp signal. In this case, an exact Generalized Likelihood Ratio (GLR) test cannot be achieved, the system leading to the Maximum Likelihood Estimates (MLE) of the parameters being non linear. The usual solutions are to derive a GLR assuming the signal piece-wise stationary or to supervise estimated parameters of the chirp. We propose two solutions taking into account both slow and fast nonstationarities of the signal. The first consists in a GLR derived from a signal phase model approximation. The second keeps the exact model and uses an approximation of the LR. Delay to the detection is studied and a discussion on estimation of the parameters after change is leaded.

1. Problem statement

This paper is concerned with detection and estimation of abrupt changes in a chirp signal. This problem is important in vibration monitoring where the measured signals reflect both the nonstationarities due to the surrounding excitation and the nonstationarities due to changes in the eigen structure. The signal model herein is:

$$y_n = A\exp(j[(\alpha/2)n^2 + \omega n + \phi]) + b_n \qquad (1)$$

$n = 0 \dots N-1$, where b_n is a white complex Gaussian noise with variance σ^2 and the parameters vector under study is $\boldsymbol{\theta} = (\alpha, \omega, \phi)$. The problem of detecting a break in $\boldsymbol{\theta}$ at instant k can be expressed by the following hypothesis test:

$$H_0: \quad \boldsymbol{\theta} = \boldsymbol{\theta}_0 = (\alpha_0, \omega_0, \phi_0) \qquad n = 0 \dots k,$$

$$H_1: \begin{cases} \boldsymbol{\theta} = \boldsymbol{\theta}_0 = (\alpha_0, \omega_0, \phi_0) & n = 0 \dots r-1, \\ \boldsymbol{\theta} = \boldsymbol{\theta}_1 = (\alpha_1, \omega_1, \phi_1) & n = r \dots k, \end{cases}$$

k going from 0 to $N-1$. H_0 is the hypothesis that no change has occurred between samples 0 and $k-1$ and H_1 is the hypothesis that a change has occurred at instant r unknown with $0 \le r \le k$.

The parameters vector before change $\boldsymbol{\theta}_0$ is assumed to be known. The instant of change r and the parameters vector after change $\boldsymbol{\theta}_1$ are unknown and have to be estimated by maximum likelihood (ML) in order to derive a generalized likelihood ratio (GLR) between conditional probability densities of the signal under each hypothesis. However ML estimate of r and $\boldsymbol{\theta}_1$ cannot be achieved, the final system being non linear.

In order to alleviate this problem, we propose, in this communication, two alternatives. The first is to derive a GLR from an approximation of the model of Eqn. (1), the second uses an approximation of the likelihood ratio from the exact model of Eqn. (1). We will see that each solution gives answers to our problem. Our purpose will be then to compare their performances.

2. Exact Generalized Likelihood Ratio with a simplified model

Assuming that the SNR, A^2/σ^2, is large, it has been proved [4] that expression (1) can be simplified as:

$$y_n = A\exp(j[(\alpha/2)n^2 + \omega n + \phi + u_n]), \quad n = 0 \dots N-1,$$

where u_n is a zero mean white Gaussian noise with variance $\sigma^2/2A^2$.

Denoting the phase of sample y_n by ψ_n, the likelihood ratio $L_1(k,r)$ between hypothesis H_0 and H_1 can be derived from ψ_n. We can show that, at the instant k, $k = 0 \dots N-1$:

$$L_1(k,r) = \sum_{n=r}^{k} \frac{A^2}{\sigma^2} (((\psi_n - (\alpha_0/2)n^2 - \omega_0 n - \phi_0)^2 - (\psi_n - (\hat{\alpha}_1/2)n^2 - \hat{\omega}_1 n - \hat{\phi}_1)^2)),$$

$\hat{\alpha}_1, \hat{\omega}_1$ and $\hat{\phi}_1$ are the MLE of $\alpha_1, \omega_1, \phi_1$ given by:

$$\begin{pmatrix} \Sigma_0 & \Sigma_1 & 2\Sigma_2 \\ \Sigma_1 & \Sigma_2 & 2\Sigma_3 \\ \Sigma_2 & \Sigma_3 & 2\Sigma_4 \end{pmatrix} \begin{pmatrix} \hat{\phi}_1 \\ \hat{\omega}_1 \\ \hat{\alpha}_1 \end{pmatrix} = \begin{pmatrix} \Sigma_{0,\psi} \\ \Sigma_{1,\psi} \\ \Sigma_{2,\psi} \end{pmatrix}, \quad (2)$$

$$\Sigma_j = \sum_{n=r}^{k} n^j, \;\; \Sigma_{j,\psi} = \sum_{n=r}^{k} \psi_n n^j. \qquad (3)$$

At instant k, the estimated value of r, $\hat{r}$, is the maximizer of $L_1(k,r)$ over $[0,k]$; this interval can be reduced, for computational cost, to a fixed length M: $[k-M+1,k]$.

Occurrence of a break is decided at the instant k when $L_1(k,\hat{r})$ overpass a fixed threshold λ:

$$\max_r \max_{\theta_1} L_1(k,r) \underset{H_0}{\overset{H_1}{\lessgtr}} \lambda \qquad (4)$$

Estimates of α, ω and ϕ are then obtained from $\hat{r}$, with eqns. (2).

3. Approximated likelihood ratio with an exact model

If the model (1) is kept, an approximated likelihood ratio called "local approach" , [1, 3], can be derived under the assumption of a relation between the parameters vector before and after change. We assume herein linear relation: $\theta^1 = \theta^0 + \lambda c$ where c is the "change direction" vector satisfying $||c||_2 = 1$. Under these hypothesis and assuming $|\lambda| = 1$, the first order approximation of the likelihood ratio is:

$$\begin{aligned} L_2(k) &= \Delta_1^{k^T}(\theta) c, \\ \Delta_1^{k^T}(\theta) &= \begin{pmatrix} \partial \log p_{\theta}(y)/\partial\alpha \\ \partial \log p_{\theta}(y)/\partial\omega \\ \partial \log p_{\theta}(y)/\partial\phi \end{pmatrix}^T, \end{aligned}$$

$p_{\theta}(y)$ is the conditional distribution of the samples vector $y = (y_0 \dots y_k)$.

Using the independence between signal and noise , denoting by c_1, c_2, c_3, coordinates of c and after some calculations, $L_2(k)$ can be expressed as the following cumulative sum:

$$L_2(k) = s_1^k(\alpha)c_1 + t_1^k(\omega)c_2 + u_1^k(\phi)c_3. \qquad (5)$$

Each term of this sum illustrates the contribution of a signal parameter:

- Sweep rate contribution:

$$s_1^k(\alpha) = -\frac{2A}{\sigma^2}\sum_{n=0}^{k} n^2 \Im(y_n^* \exp j([(\alpha/2)n^2 + \omega n + \phi])),$$

- Pulsation contribution:

$$t_1^k(\omega) = -\frac{2A}{\sigma^2}\sum_{n=0}^{k} n \Im(y_n^* \exp j[(\alpha/2)n^2 + \omega n + \phi])),$$

- Initial phase contribution:

$$u_1^k(\phi) = -\frac{2A}{\sigma^2}\sum_{n=0}^{k} \Im(y_n^* \exp j([(\alpha/2)n^2 + \omega n + \phi])),$$

Occurrence of a break is decided if $L_2(k)$ overpass a fixed threshold λ:

$$L_2(k) \underset{H_0}{\overset{H_1}{\lessgtr}} \lambda \qquad (6)$$

The instant of break, $\hat{r}$, is taken as the instant of detection. The jump magnitude can be recovered by an additional procedure. In effect, under the assumption of large SNR, it can be demonstrated that for $k \le r-1$, $s_1^k = t_1^k = u_1^k \approx 0$ and for $k \ge r$:

$$\begin{aligned} s_1^k &\approx \frac{2A}{\sigma^2}\sum_{n=r}^{k} n^2 \sin(((\alpha_1-\alpha_0)/2)n^2) &(7)\\ t_1^k &\approx \frac{2A}{\sigma^2}\sum_{n=r}^{k} n \sin((\omega_1-\omega_0)n) &(8)\\ u_1^k &\approx \frac{2A}{\sigma^2}\sum_{n=r}^{k} \sin(\phi_1-\phi_0) &(9) \end{aligned}$$

This allows the derivation of a sweep rate estimation procedure from samples $(s_1^r, s_1^{r+1}, s_1^{r+2}, \dots)$ or on the samples of the signal: $(y_r, y_{r+1}, \dots, y_{r+L-1})$, where L should be sufficiently but not too large to make a compromise between accuracy of the estimation and delay for the next detection.

4. Experimental results

The signal used is a noisy chirp showing a break in the sweep rate: $\theta_0 = (0.01252, 0.6912, 0)$, $\theta_1 = (0.0112, 0.6912, 0)$, $r = 45$, $N = 100$.

Fig. 1 represents the real part of the signal and its argument, i.e. ψ_n for a snr equal to $10dB$. Fig. 2 shows the result of the detector 1, i.e. $L_1(k,\hat{r})$. Fig. 3 represents the result of detector 2, i.e. $L_2(k)$ for $c = (-1,0,0)$. We can note that $L_1(k,\hat{r})$ does not equal zero before the change since $\hat{\alpha}_1$ does not converge to α_0 (see Fig. 4). The reason is that the search over r is performed on a fixed window, $M = 10$ samples.

4.1. Delay to the detection

We study in this section delay to the detection as a function of false alarm rate.

Usually, a detection problem consists in determining probability of fixed delay to the detection, Pd(r) where r is the delay, and probability of false alarm, Pfa, relatives to the detector. The problem is to choose a threshold λ which ensures a fixed false alarm rate whatever is the value of the snr. Analytic expressions

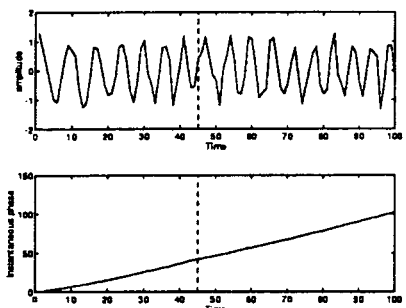

Figure 1: Real part and argument of y_n.

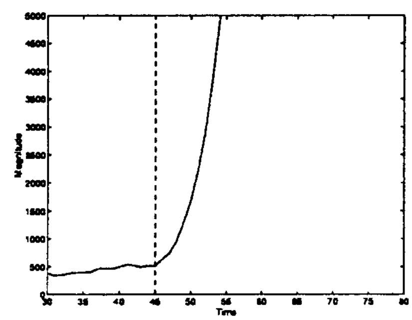

Figure 2: Result of detector 1: $L_1(k, \hat{r})$.

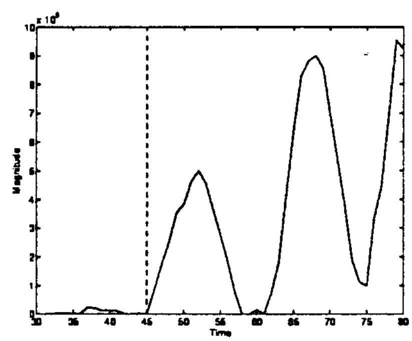

Figure 3: Result of detector 2: $L_2(k)$.

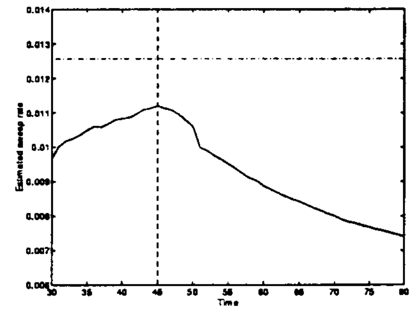

Figure 4: $\alpha_1(k)$ estimated from $L_1(k, \hat{r})$.

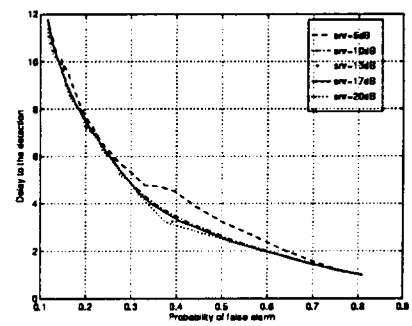

Figure 5: Delay to the detection as a function of false alarm rate for the detector 1:$L_1(k, \hat{r})$

of Pfa and Pd(r) function of λ require analytic expression of the detector distribution under H_0 and H_1 but result of the local approach has been proved to have only asymptotic properties. However, the threshold has been fixed in the following manner:

$$\lambda = \frac{1}{L} T \sum_{l=q+1}^{q+L} d_{k-l}, \qquad (10)$$

where T determines Pfa and $\frac{1}{L}\sum_{l=q+1}^{q+L} d_{k-l}$ is an estimation of the mean of d_k, over a window of fixed length: L, d_k being the result of each detector, 4,6. q are the number of "guard samples". It is verified that curves of Pfa function of T are independent of the snr.

The signal used is just the same, 500 noise realizations have been used for each snr.

Fig. 5 shows delay to the detection as a function of false alarm rate for snr going from $5dB$ to $20dB$ in the case of the detector 1. Delay to the detection decreases as probability of false alarm increases, minimum delay to the detection (equal to one sample) is reached for a Pfa greater than 0.8. For "correct" values of Pfa (greater than 0.2 and less than 0.5), delay

snr(dB)	0	5	10	13	17	20
Pfa(%)	0.32	0.27	0.14	0.11	0.06	0.06

Table 1: Limits of probability of false alarm for the detection for the detector 2

to the detection is meanly equal to 5 samples. A Pfa of 0.35 seems to set a border: from 0.35, delay to the detection decreases more slowly. An important result is that performance is relatively independent from the snr. Table 1 resumes performance of the second detector. Values given are values of Pfa below which there is no detection, beyond these values the delay to the detection is minimum and equal to 1 sample.

4.2. Jump estimation

The *GLR* structure allows to estimate at the same time the vector parameters after change and the instant of break. The accuracy of the jump estimation i.e θ_1 depends on the accurate estimation of $\hat{r}$ and on the number of samples used for the estimation, see eqs. (2,3). The local approach does not permit to estimate the instant of break and the amplitude of jump but its formulation leads to approximate expressions of the detector after the jump, see eqs. (7,8,9). These results are very useful in the case of a jump in the initial phase since the problem of jump estimation is reduced to estimation of the slope of the detector after jump. In the case of a frequency jump or a sweep rate jump, the problem remains the same: estimation of a frequency or estimation of a sweep rate. In this last case, estimating the new sweep rate on the signal after the instant of detection is better than estimating the magnitude of jump on the result of the detector, see fig. (3), for two mains reasons: first a treatment on the samples $(s_r^k, s_r^{k+1}, \ldots)$ is necessary to get samples of a linearly modulated sinusoid (eq. 7) and second the noise on s_1^k is no more Gaussian and white for low snr.

Fig. (6) shows estimated value of $\alpha_1(k)$ for values of Pfa going from 0.1 to 0.9, the true value being 0.012. For a *snr* of $5dB$, $\hat{\alpha}_1$ is fluctuating around a value of 0.0103. For all others snr going from $10dB$ to $20dB$, curves are almost superimposed and reach a fixed value of 0.011 from a Pfa equal to 0.35, value which is also a limit for the delay to the detection, see fig. (5). Some remarks more or less expected can be made from these results. First, the estimation of α_1 improves for growing Pfa going from 0.1 to 0.35, from which it stays to a fixed value. This result can be explained by looking at the delay to the detection, when delay to the detection is too high, neither estimated value of r nor estimation of α_1 are correct. Second, from a sufficient

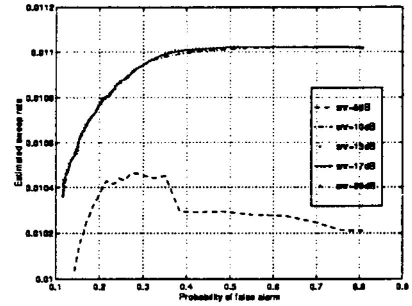

Figure 6: Estimated sweep rate as a function of false alarm rate for the detector $1{:}L_1(k,\hat{r})$

value of Pfa, delay to the detection is minimum and estimation of α_1 improves but shows a bias due in part to the fixed search window of $\hat{r}$.

5. Conclusion

Two solutions to detect and estimate a jump in the parameters of a chirp signal have been presented in this paper. The first one uses an exact GLR with a signal phase model of the signal and allows estimation of the parameters after change. The second one keeps the exact model of the chirp signal with an approximated LR called local approach and requires an external procedure for the jump estimation. Delay to the detection for the GLR decreases as the false alarm rate increases, there is no detection at all for very low false alarm. The local approach has a fixed and minimum delay to the detection over a fixed false alarm rate which depends on the *snr*. Use of one or the other algorithm will then depends on the assigned constraint for the application under study.

6. References

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