ITERATIVE CONSTRAINED LEAST SQUARES ARRAY SHAPE ESTIMATION USING SOURCES IN UNKNOWN LOCATIONS UNDER A PLANEWAVE ASSUMPTION

Jonathan H. Gross and David E. Grant
University of Texas Applied Research Laboratories
10000 Burnet Road
Austin, TX 78713-8029 USA

ABSTRACT

Array shape estimation is an important problem in array signal processing. We present an iterative algorithm that estimates two dimensional array element positions via an algebraic solution to a least squares problem, under the assumption that the sources are in the far-field with unknown positions. Under some assumptions, we calculate the dependence of performance on the distribution of the localizing sources. We give examples of algorithm performance from both simulations and real data. We also illustrate algorithm performance for horizontal arrays as a function of time delay estimation variance and the variance of vertical arrival angle estimates.

1. INTRODUCTION

Array shape estimation is an important problem in array signal processing. Without adequate knowledge of sensor positions, array gain losses due to mismatch for adaptive beamforming [1] and sidelobe levels for conventional beamforming can be large. Super resolution algorithms are also highly sensitive to errors in sensor positions.

A number of algorithms for array shape estimation have been introduced in recent years. Rockah and Schultheiss [2] have examined the sensor location uncertainty problem in detail, presenting an algorithm for array shape estimation using disjoint sources, which can be separated in either time or frequency, and deriving the Cramer-Rao lower bound on the variance of the estimated sensor positions. Weiss and Friedlander [3] have discussed an iterative algorithm for array shape estimation, using estimated sensor positions to obtain a maximum likelihood estimate of the arrival angles of the incident energy from the localizing sources, and using these arrival angle estimates to obtain an estimate of each sensor position which minimizes an objective function. Lo and Marple have suggested an eigenstructure based method for array shape estimation [4] when the directions of sources are known, and have also addressed observability conditions for the array shape estimation problem [5].

In this paper, we introduce an iterative array shape estimation algorithm based on constrained least squares. The method is similar in nature to [3] but estimates sensor positions via an algebraic solution of a constrained least-squares problem. We assume that array sensor separations are physically constrained to be less than or equal to the nominal values as is often the case in, for example, cabled arrays of hydrophones for sonar signal processing. We also relax the assumption of sources being co-planar with the array of sensors. Array shape information is obtained from sources of opportunity with no assumptions of temporal or spectral disjointness.

2. ALGORITHM FORMULATION

Let the distance in the direction of source k from sensor 1 to sensor $i+1$ be denoted as d_{ki}. That is, place sensor 1 at the origin, and let d_{ki} be defined as $d_{ki} = \mathbf{s}_k^T \mathbf{r}_i$, where $\mathbf{s}_k$ is the direction vector for the k^{th} source and $\mathbf{r}_i$ is the position vector for sensor $i+1$. Assuming that all sources being used are in the far-field, so that a planewave assumption is valid, these distances are related to the sensor positions and directions of arrival (DOA) as

$$d_{ki} = x_{i+1}\sin(\theta_k)\cos(\phi_k) + y_{i+1}\cos(\theta_k)\cos(\phi_k) \qquad (1)$$

where θ_k is the DOA of energy from source k, in the x-y plane relative to the positive y axis, and ϕ_k is the DOA above the x-y plane, or the vertical arrival angle. Given n sensors and p sources, this relationship results in a set of linear equations given by

$$\mathbf{D} = \mathbf{\Phi} \cdot \mathbf{A} \qquad (2)$$

where

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1(n-1)} \\ d_{21} & d_{22} & & d_{2(n-1)} \\ \vdots & & \ddots & \\ d_{p1} & d_{p1} & \cdots & d_{p(n-1)} \end{bmatrix} \qquad (3)$$

$$\mathbf{\Phi} = \begin{bmatrix} \sin(\theta_1)\cos(\phi_1) & \cos(\theta_1)\cos(\phi_1) \\ \sin(\theta_2)\cos(\phi_2) & \cos(\theta_2)\cos(\phi_2) \\ \vdots & \vdots \\ \sin(\theta_p)\cos(\phi_p) & \cos(\theta_p)\cos(\phi_p) \end{bmatrix} \qquad (4)$$

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$$A = \begin{bmatrix} x_2 & x_3 & \cdots & x_n \\ y_2 & y_3 & \cdots & y_n \end{bmatrix} \tag{5}$$

Using an estimate of **D**, denoted as $\hat{\mathbf{D}}$, and obtained from the data via crosscorrelation between sensor pairs, or alternative techniques, and an estimate of $\mathbf{\Phi}$, denoted as $\hat{\mathbf{\Phi}}$, a quadratically constrained least-squares problem may be formulated as

$$\min_{\hat{A}_i} \left\| \hat{\mathbf{\Phi}}\hat{\mathbf{A}}_i - \hat{\mathbf{D}}_i \right\|_2^2 \quad \text{subject to} \left\| \hat{\mathbf{A}}_i - \hat{\mathbf{A}}_{i-1} \right\|_2 \le \alpha_i \tag{6}$$

where the subscript i denotes the i^{th} column, and α_i is the nominal distance between sensor $i+1$ and sensor i. Equation (6) can be solved via singular value decomposition and the method of Lagrange multipliers [6].

We find the arrival angles which minimize the error between the observed distances and those calculated from the current estimate of sensor positions by solving

$$\min_{\theta_k, \phi_k} \left\| \hat{\mathbf{A}}^T (\hat{\mathbf{\Phi}}^T)_k - (\hat{\mathbf{D}}^T)_k \right\|_2^2 \tag{7}$$

for $k = 1, 2, \ldots, p$, under the assumption of either known or unknown vertical arrival angles. Note that (7) is nonlinear in θ_k, ϕ_k, but can be solved via a simple one or two dimensional search (depending on whether ϕ_k is assumed known), or other more efficient techniques. In some cases, vertical arrival angles may be assumed known, as in some underwater acoustics problems where the sound speed profile and bottom properties result in an asymptotic vertical arrival angle for long range sources. After arrival angles are estimated, a new set of sensor positions is calculated using (6). The above two steps are repeated until convergence is obtained. Convergence is obtained when the fit to $\hat{\mathbf{D}}$ goes essentially unimproved, that is

$$\left\| \hat{\mathbf{D}} - \hat{\mathbf{\Phi}}(m-1)\hat{\mathbf{A}}(m-1) \right\| - \left\| \hat{\mathbf{D}} - \hat{\mathbf{\Phi}}(m)\hat{\mathbf{A}}(m) \right\| < \varepsilon \tag{8}$$

where $\hat{\mathbf{A}}(m)$ and $\hat{\mathbf{\Phi}}(m)$ denote the estimates of **A** and $\mathbf{\Phi}$ at iteration m.

3. SIMULATION EXAMPLE

Figure 1 shows the nominal and actual array element positions as a percentage of nominal array length for a 10 element tapered array. The actual array is bowed with peak bow displacement equal to 10% of the array's aperture. The integrated array length is 97 percent of nominal, indicating that the array is also contracted by 3 percent. Figure 2 shows the error in estimated element positions versus iteration number for the constrained algorithm and for the unconstrained algorithm ($\alpha_i = \infty$ in (6)). For this simulation, we let the d_{ki} estimates have a Gaussian distributed error with zero mean and a standard deviation of 0.9% of the average nominal element spacing. We also let the vertical arrival angles be Gaussian distributed with a mean of 10° above horizontal, and a standard deviation of 5°. Thirty sources were used, distributed uniformly [0°,180°] over azimuth. Note that the constrained algorithm converges faster than the unconstrained algorithm.

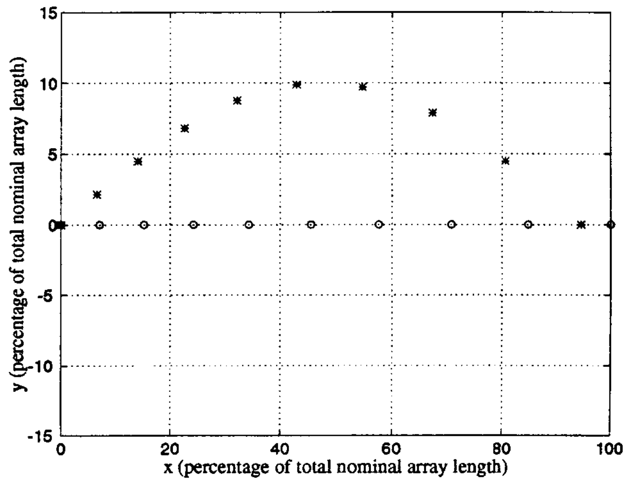

FIG. 1 Nominal (o) and actual (*) array configurations for simulation example.

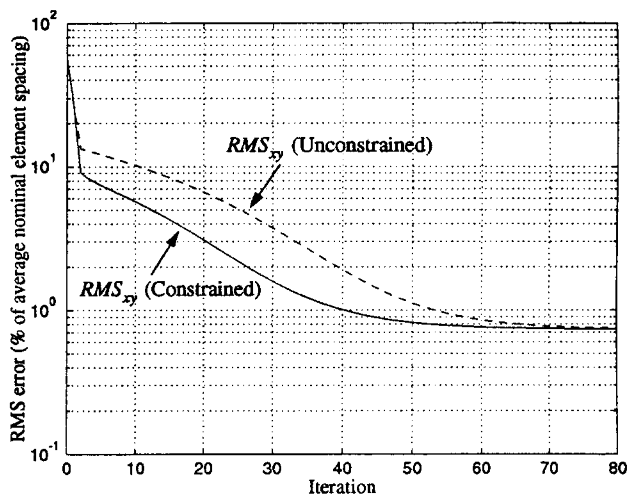

FIG. 2 RMS error in array element position estimates, as a percentage of average nominal element spacing, vs. number of iterations for simulation example.

4. DATA EXAMPLE

Figure 3 shows the nominal, actual, and estimated array element positions as a percentage of total nominal array length for an underwater bottom mounted horizontal line array. Crosscorrelation between sensor pairs was used to generate an estimate of **D**, using 16 source samples selected from the correlagrams. The vertical arrival angle was assumed to be 10° above horizontal. The RMS difference between the estimated and actual positions is 0.85% of the average nominal element spacing. (The actual positions were obtained via a high-accuracy technique which utilizes known source positions.)

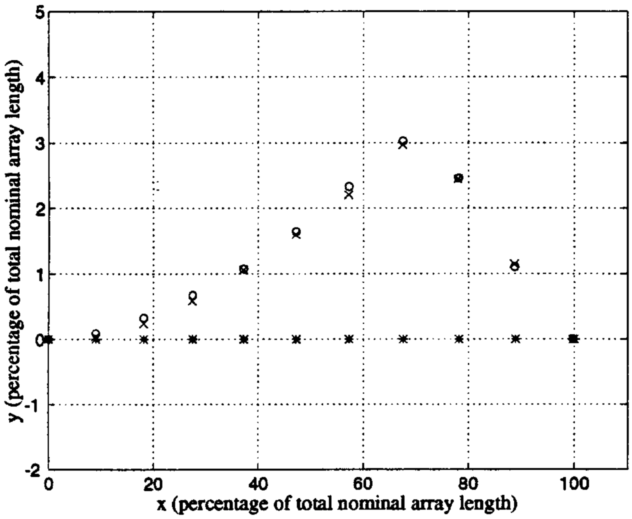

FIG. 3 Nominal (*), actual (o), and estimated (x) array element positions for underwater acoustic array example.

5. ERROR ANALYSIS

In practice, measurements of **D** from the data will be corrupted by noise. The appropriate linear model is then

$$\hat{\mathbf{D}} = \mathbf{\Phi} \cdot \mathbf{A} + \mathbf{N} \quad (9)$$

where **N** is a noise matrix, representing noise in the estimate of **D**. Given the arrival angle matrix, $\mathbf{\Phi}$, the least squares solution for each column of **A** in the absence of active constraints may be expressed as [6]

$$\hat{\mathbf{a}} = \mathbf{\Phi}^{\dagger}\hat{\mathbf{d}} \quad (10)$$

where $\mathbf{\Phi}^{\dagger} = (\mathbf{\Phi}^T\mathbf{\Phi})^{-1}\mathbf{\Phi}^T$, and $\hat{\mathbf{d}}$ is the appropriate column of $\hat{\mathbf{D}}$.

If **n**, a column of **N**, is assumed Gaussian distributed as $N(0, \sigma_n^2\mathbf{I})$, then the least squares solution for **a**, given by (10), is unbiased and Gaussian distributed with covariance matrix given by [7]

$$\text{cov}(\hat{\mathbf{a}}) = \sigma_n^2(\mathbf{\Phi}^T\mathbf{\Phi})^{-1} \quad (11)$$

The Gaussian assumption on the distribution of **N** is reasonable for cross-correlation based time delay estimates assuming uncorrelated Gaussian signal and noise at high SNR [8]. The variance of the error in element positions (for both x and y, i.e. $\sigma_{xy}^2 = \sigma_x^2 + \sigma_y^2$) may then by quantified as

$$\sigma_{xy}^2 = \text{trace}\left[(\mathbf{\Phi}^T\mathbf{\Phi})^{-1}\right] \quad (12)$$

The trace of $(\mathbf{\Phi}^T\mathbf{\Phi})^{-1}$ may be written as

$$\text{trace}\left[(\mathbf{\Phi}^T\mathbf{\Phi})^{-1}\right] = \frac{1}{\det(\mathbf{\Phi}^T\mathbf{\Phi})}\text{trace}(\mathbf{\Phi}^T\mathbf{\Phi}) \quad (13)$$

where

$$\text{trace}(\mathbf{\Phi}^T\mathbf{\Phi}) = \sum_{i=1}^{p}\cos^2(\phi_i) \quad (14)$$

and

$$\det(\mathbf{\Phi}^T\mathbf{\Phi}) = \sum_{i=1}^{p-1}\sum_{k=i+1}^{p}\cos^2(\phi_i)\cos^2(\phi_k)\sin^2(\theta_i - \theta_k) \quad (15)$$

Note from (15) that for finite variance in the estimated element positions, the arrival angles of the localizing sources must be different (see also [5]). Figure 4 shows a plot of element position estimate variance versus the width of the bearing sector over which the sources are uniformly distributed for 10, 20, and 50 sources (vertical arrival angles were 10°). Note that as the number of sources increases, or the width of the bearing sector over which the sources are uniformly distributed increases, the variance in the element position estimates decreases.

6. NUMERICAL PERFORMANCE EVALUATION

We numerically evaluate algorithm performance, for the array configuration of Figure 1, with respect to variance in time delay estimation noise (or equivalently variance in d_{ki} estimates, given by σ_n^2) and variance in the error between assumed and actual vertical arrival angles, or σ_ϕ^2. We assume time delay estimation errors are independently Gaussian distributed with zero mean for each source. We also assume vertical arrival angles to be Gaussian distributed, with known mean. We evaluate RMS error in estimated element positions using 50 realizations for each scenario of interest. Figure 5 shows the mean RMS error in element positions versus time delay estimation variance, assuming $\sigma_\phi^2 = 0$, for p=10, 20 and 50. Figure 6 illustrates RMS error versus vertical arrival angle variance (σ_n = 0.9% of average nominal element spacing) assuming that the mean vertical arrival angle of 10° is known.

7. CONCLUSIONS

We have introduced a constrained least squares algorithm for array shape estimation using sources in unknown locations under a planewave assumption. The algorithm is demonstrated using both simulation and real data examples. Performance is analyzed in the absence of arrival angle errors as a function of the number and distribution of the localizing sources. Performance is also characterized numerically as a function of the number of localizing sources, time delay estimation noise, and noise in vertical arrival angle estimates.

ACKNOWLEDGMENT

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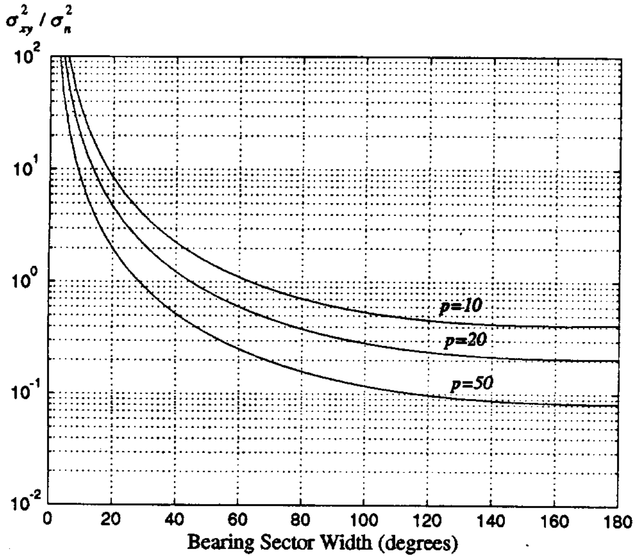

FIG. 4 Ratio of variance in estimated element positions to d_{ki} variance as a function of source bearing sector width.

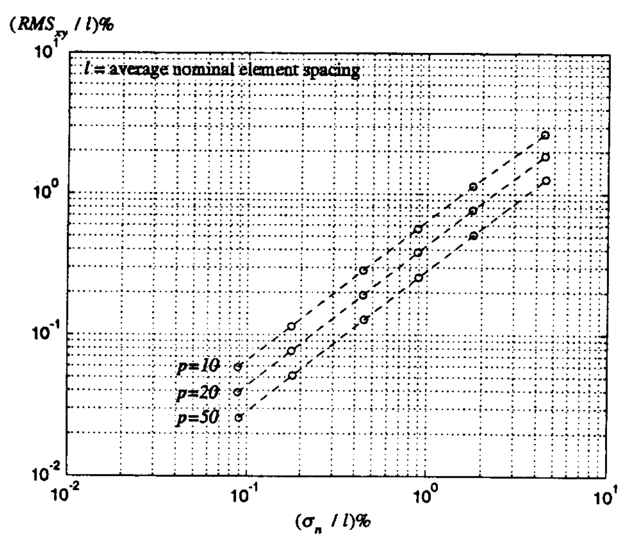

FIG. 5 RMS error in array element position estimates vs. standard deviation of error in d_{ki} estimates, σ_n, ($\sigma_\phi = 0$).

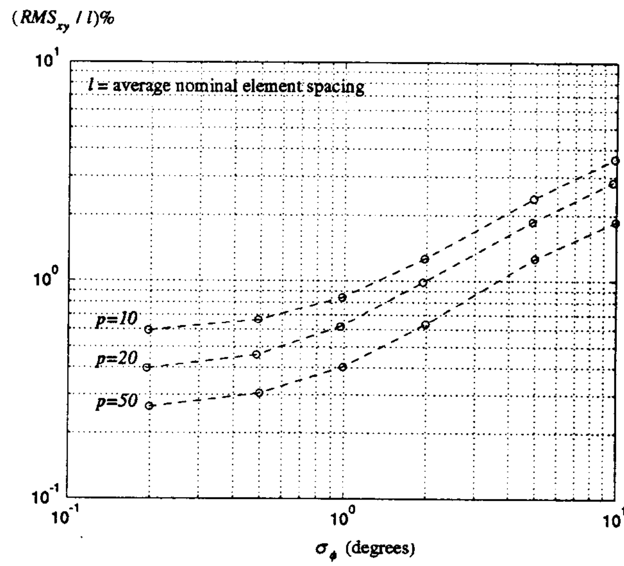

FIG. 6 RMS error in array element position estimates vs. standard deviation in vertical arrival angles, σ_ϕ, ($\sigma_n / l = 0.9\%$).

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