Unified Approach to Snakes, Elastic Nets and Kohonen Maps

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Abstract

Snakes, elastic nets and Kohonen networks are well known algorithms which were developed in different contexts. However, these algorithms share common features allowing us to ask what is the relationship among them and suggesting their use in problems which have traditionally been tackled by only one of them. This paper addresses the problem of edge linking and proposes a new class of non-linear recursive algorithms, based on a general cost function, which includes snakes, Kohonen maps, and elastic nets as special cases. This class provides an unified framework for several existing algorithms in Pattern Recognition and Active Contours and allows the design of new recursive schemes.

1. Introduction

Edge linking is a basic operation in many image analysis and computer vision systems. The goal of an edge linker can be stated as follows: given a set of edge points, computed by some edge detection algorithm, try to find a continuous path through these points which approximates the object boundary. In general, this is a hard task since the algorithm does not know the shape of the object, or its pose, and it has to choose among several possible paths, fill the gaps and eliminate spurious edges.

Edge linking shows a close resemblance with problems studied in different contexts. e.g., the traveling salesman problem or data ordering problems solved by auto-associative neural networks, and constrained clustering problems. Therefore, we can legitimately ask whether the algorithms used to solve these problems (e.g., elastic nets and Kohonen maps) can be successfully applied to edge linking.

In this paper, three recursive algorithms are considered: the snake active contours proposed by Kass et al. in 87 [1] in the context of shape analysis and two well known neural networks (the elastic net [2] and the self-organizing feature map or Kohonen map [3]) proposed in different contexts. The snake is a deformable model consisting of a set of units, each of them associated with a point in the image, which are attracted in each iteration by image features (e.g., edge points) and acted by internal (*regularization*) forces which try to keep some coherence during the training process. The shape of the snake model is randomly initialized and it is then recursively modified through an iterative process until it reaches a final configuration and hopefully approximates the object boundary.

Similar principles can also be found in the context of the elastic nets proposed by Durbin et al. in 87 [2] and in the topological map of patterns proposed by Kohonen in 82 [3]. In fact, both algorithms also approximate the input data by a sequence of points or weighting vectors, initialized with some configuration, and then recursively updated using specific update rules.

Two basic questions are addressed in this paper: *(i) what is the relationship among these three algorithms ? (ii) can they be used in the context of object linking ?*

To answer the first question we show in this paper that the three algorithms, with minor modifications, are special cases of a more general update rule and solutions of a general optimization problem. Therefore, we propose a new class of non-linear active contour algorithms based on a general update law, which includes snakes, Kohonen maps, and elastic nets as special cases. The usefulness of this class of algorithms is twofold: it provides an unified framework for several existing algorithms in Pattern Recognition and Active Contours and allows the design of new recursive schemes (e.g., hybrid schemes).

2. Previous Algorithms

Let us denote by p=x+jy the location of an edge point in an image, and let $P=\{p(1),\dots,p(N)\}$, $p(n)\in\mathbf{C}$, be the set of all edge points detected in the image by some edge detection algorithm. The goal of an edge linking operation is to approximate the edge points contained in P by a 1D model consisting of an ordered set of units, $z_k\in\mathbf{C}$, $k=1,\dots,M$. For convenience, we assume that the model units are organized in a complex vector $\mathbf{z}=[\,z_1,\dots,z_M\,]^T$.

2.1 Snakes

One popular technique for object boundary detection which has been extensively studied during these last seven years is the snake active contour algorithm proposed by Kass et al. in [1]. The choice of the snake shape, $\mathbf{z}\in C^M$, is formulated as an nonlinear optimization problem defined by the cost function, [4]

$$J=\mathbf{z}^H\mathbf{A}_s\mathbf{z}-\sum_{k=1}^{M}\sum_{p\in P}\phi_\sigma\left(\left|z_k-p\right|^2\right) \qquad (1)$$

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where $\phi_\sigma(d) = \exp(-d/2\sigma^2)$, $|.|$ denotes the modulo operation and $\mathbf{A}_s$ is a MxM real, pentadiagonal, Toeplitz matrix. The optimization of (1) is usually performed by using the steepest descent algorithm,

$$\mathbf{z}^{t+1} = \mathbf{z}^t - \gamma \nabla_\mathbf{z} J \tag{2}$$

where γ is the learning step and $\nabla_\mathbf{z}$ is the complex gradient operator [5]. Using (1,2) we obtain a recursive update law for the snake model [1]

$$\mathbf{z}^{t+1} = \mathbf{z}^t - \gamma \mathbf{A}_s \mathbf{z}^t + \gamma \mathbf{f}_{ext}^t \tag{3}$$

where $\mathbf{f}_{ext} = [\, f_{ext}(z_1), \ldots, f_{ext}(z_M)\,]^T$ is a complex vector containing the external forces acting in each snake unit which are samples of an external force field

$$f_{ext}(z) = \frac{1}{\sigma^2} \sum_p \, (p - z)\, \phi_\sigma(|z - p|^2) \tag{4}$$

2.2 Elastic Nets

In 1987 Durbin and Willshaw introduced the elastic net concept to find short routes for the traveling salesman problem (TSP) [2]. Unlike combinatorial methods which have traditionally been used to solve this problem, elastic nets search the optimal solution in a continuous space of trajectories using soft constraints. Elastic nets have also been used in image analysis (e.g., in hand-printed digit recognition [6]) and may also be interpreted as a deformable model for object boundary extraction. In fact, if we replace the cities of the TSP by edge points detected in an image, the elastic net algorithm will try to link the detected points with a trajectory of minimal length. Although edge linking and TSP seem similar, there are however some significant differences: (i) one of the goals of an edge linker is to discard spurious edges, while the TSP problem assumes noiseless data, and therefore forces the trajectory to visit all cities (even those which are far away); (ii) edge linking usually penalizes long paths and high curvature strokes, while in the TSP only the first constraint is considered. These differences yield a different cost function [2],

$$J = \mathbf{z}^H \mathbf{A}_{en} \mathbf{z} - \sum_p \log \sum_{k=1}^{M} \phi_\sigma\left(\left|z_k - p\right|^2\right) \tag{5}$$

where $\mathbf{A}_{en}$ is a regularization matrix similar to $\mathbf{A}_s$ (however, in the TSP $\mathbf{A}_{en}$ is chosen to be tridiagonal instead of pentadiagonal allowing high curvature points). Applying the steepest descent algorithm to (5), one obtains the recursion

$$\mathbf{z}^{t+1} = \mathbf{z}^t - \gamma \mathbf{A}_{en} \mathbf{z}^t + \gamma \mathbf{f}_{ext}^t \tag{6}$$

which has the same structure as (3), but with the external force, given by

$$f_{ext}(z) = \frac{1}{\sigma^2} \sum_p \frac{(p - z)\phi_\sigma(|z - p|^2)}{\sum_{k=1}^{M} \phi_\sigma(|z_k - p|^2)} \tag{7}$$

In elastic nets, the structure of the external force field is more complex than in snakes, (see eq. (4)) since the force applied to each unit depends on the location of the remaining units. Elastic nets can be therefore interpreted as a competitive learning technique.

2.3 Kohonen Maps

Another iterative scheme which may also be used in the context of edge linking is the topological map of patterns, proposed by Kohonen in [3]. This technique has been used in a wide range of problems, including TSP. In the case of unidimensional maps, the relationship between Kohonen maps and snakes has already been addressed by us in [7]. In fact, the Kohonen algorithm shares two important features with snakes and elastic nets: (i) the three are unsupervised learning techniques where the model units are attracted toward the locations of the detected edges during the training process, and (ii) they organize the model units into 1D sequence, where neighboring units are usually near in the image.
In the context of object boundary detection, we shall assume that the input patterns are the coordinates of 2D edge points, and the Kohonen map is chosen to be a unidimensional set of units.
To train the Kohonen map, each input pattern is sequentially presented to the network. For each pattern, the algorithm selects the nearest unit, z_{k_a}, usually denoted by *active cell*, where

$$k_a = \arg\min_k |p - z_k| \tag{8}$$

The network is then updated using the following rule[1]

$$z_n^{t+1} = z_n^t - \gamma\left(p - z_n^t\right) \qquad n = k_a - \beta, \ldots, k_a + \beta \tag{9}$$

where β is the neighborhood radius and γ is the learning step as before.
Alternatively, the network can be trained in a batch mode [7]. This can be accomplished by freezing the network state during a whole epoch (i.e., during the presentation of all edge coordinates) and performing the update of the contour units at the end of each epoch. In this case, one also obtain an update law of the form,

$$\mathbf{z}^{t+1} = \mathbf{z}^t + \gamma \mathbf{f}_{ext}^t \tag{10}$$

where the external force components are given by[2],

$$f_{ext}(z_k) = \sum_{n=k-\beta}^{k+\beta} \sum_{p \in P_n} \left(p - z_k^t\right) \tag{11}$$

where $P_n \subset P$ denotes the set of input patterns which choose the n-th unit as their nearest unit and β is the neighborhood radius as before. The choice of the nearest unit creates a partition of the

[1] in closed contour models, the first map unit is assumed to be neighbor of the first.

2 strickly speaking, $\mathbf{f}_{ext}$ also accounts for regularization effects since it depends on the contour configuration.

complex plane into disjoint regions which are denoted *Voronoi cells* [8]. Equations (10,11) can be interpreted as the minimization of the cost function

$$J = \sum_{k=1}^{M} \sum_{\substack{n=k-\beta \\ n \neq k}}^{k+\beta} \sum_{p \in P_n} |p - z_k|^2 + \sum_{k=1}^{M} \sum_{p \in P_k} |p - z_k|^2 \qquad (12)$$

by the steepest descent algorithm. We have split this cost function into two parts to emphasize the existence of a regularization term and an external potential, as in the previous algorithms.

3. Common Framework

This section presents a class of nonlinear recursive algorithms which contains snakes, elastic nets and Kohonen maps as special cases and allows the design of new recursive schemes. To accomplish this goal we shall develop a common update rule derived from a general cost function which has two terms: a quadratic regularization term and a clustering term defined as a weighted sum of the distances between input patterns and model units, i.e

$$J = \mathbf{z}^H \mathbf{A} \mathbf{z} + \sum_p \sum_{k=1}^{M} w_k(p) d_k(p) \qquad (13)$$

where $\mathbf{z}$ is the contour model, $\mathbf{A}$ is a real MxM matrix, $d_k(p)$ is the distance of pattern $p \in P$ to the location of the k-th unit and $w_k(p)$ is a weighting function. Cost function (13) can be considered as an extension of the distortion measures used in classic clustering and vector quantization methods (e.g., k-means, or Lloyd-Max algorithm [8]) the difference being the use of weighting factors and a quadratic regularization term.

Two hypothesis will be assumed in the sequel: i) the distance between input patterns and model units is the squared Euclidean metric

$$d_k(p) = |p - z_k|^2 \qquad (14)$$

ii) the weighting function of pattern p with respect to the k-th unit depends on the distance from p to all units of the model

$$w_k(p) = f_k(d_1, \ldots, d_M) \qquad (15)$$

and verifies a normalization condition $w_k(z_k) = 1$.

Under these hypothesis, the cost function is completely specified by matrix $\mathbf{A}$ and by the choice of the weighting functions. Let us now derive an update law for the minimization of (13) by the gradient algorithm.

<u>**Fact 1**</u>

The complex gradient of the cost function (13) with respect to z_m is given by (see Appendix).

$$\nabla_{z_m} J = 2(\mathbf{A}\mathbf{z})_m + 2\sum_p (z_m - p) \vartheta_m(p) \qquad (16)$$

where

$$\vartheta_k(p) = w_k(p) + \sum_{j=1}^{M} d_j(p) \frac{\partial w_j(p)}{\partial d_k(p)} \qquad (17)$$

Replacing (16) in the expression of the gradient algorithm

$$\mathbf{z}^{t+1} = \mathbf{z}^t - \gamma \mathbf{A} \mathbf{z}^t + \gamma \mathbf{f}_{ext}^t \qquad (18)$$

$$f_{ext}(z_k) = \sum_p (p - z_k) \vartheta_k(p) \qquad (19)$$

which defines a class of nonlinear recursive schemes for constrained clustering and edge linking. In order to define a new algorithm belonging to this class all we have to do is to specify: i) matrix $\mathbf{A}$, which can be interpreted as a lowpass filtering operation; and ii) the functions $w_k(p)$ which define the external potential in (13) or the functions $\vartheta_k(p)$ which define recursion (18,19).

Comparing the cost functions of snakes, elastic nets and Kohonen maps with (13), it can easily be shown that these algorithms are special cases of recursion (18,19) and cost function (13) which are obtained by a proper choice of $w_k(p)$ and $\vartheta_k(p)$ as shown in Table I.

	$\mathbf{w_k(p)}$	$\boldsymbol{\vartheta_k(p)}$
Snakes	$2\sigma^2 \frac{1-\phi_\sigma(d_k)}{d_k}$	$\phi_\sigma(d_k)$
Elastic Nets	$-2\sigma^2 \frac{\log \sum_{j=1}^{M} \phi_\sigma(d_j)}{d_k}$	$\frac{\phi_\sigma(d_k)}{\sum_{j=1}^{M} \phi_\sigma(d_j)}$
Kohonen Maps	1 if $\lvert k - k_a \rvert \leq \beta$; 0 otherwise	1 if $\lvert k - k_a \rvert \leq \beta$; 0 otherwise

Table 1 - Weighting functions for snakes, elastic nets and Kohonen maps

4. Interpretation

The external force field of the common framework (19) suggests several interpretations, e.g., a physical interpretation: if we assume that every input pattern, $p \in P$, is connected with the k-th unit by a spring with constant $\vartheta_k(p)$, the external force applied on the k-th unit is the sum of all spring forces. Instead, using a fuzzy terminology $\vartheta_k(p)$ can be interpreted as a degree of membership of the pth pattern to the k-th unit.

Let us define *attraction region* of the k-th unit as the set of points $p \in C$ such that $\vartheta_k(p)$ is greater than a given threshold. The attraction region of the k-th unit can be interpreted as the locus of input patterns which produce a significant contribution to the external force applied to the unit.

Figure 1 shows the attraction regions of the three algorithms using a small contour model with six units. The attraction regions were computed from the weights $\vartheta_k(p)$ defined in Table I, assuming that pattern p takes all possible values in the image and verifies the condition $\vartheta_k(p) > 0.1$.

It can be concluded from Fig. 1i that in snakes the attraction regions are circles and they do not overlap except if the units are close together. In this case, the same patterns can attract more than one model unit making them collapse. In Kohonen maps, the attraction regions are nonoverlaping Voronoi cells [8] which define a partition of the image (see Fig. 1iii). Finally, the attraction regions of elastic nets resemble the Voronoi cells of the Kohonen maps (and they tend to the Voronoi cells as $\sigma \to 0$) but they have some overlap (see Fig. 1ii). The structure of the attraction regions allows us to classify snakes as a *local* method and elastic net and Kohonen maps as *global.*

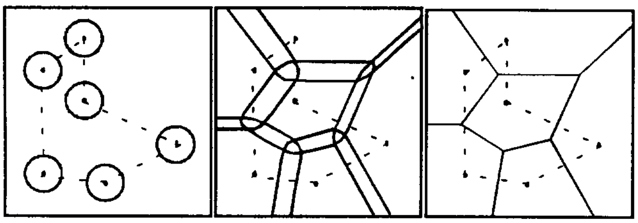

Figure 1 - Attraction Regions in (i) Snakes; (ii) Elastic Nets with and (iii) Kohonen Maps

5. Experimental Results

Figure 2 shows the performance of the three algorithms in an typical edge linking operation, allowing us to identify some of the difficulties of each algorithm. Many of these difficulties can be alleviated by the design of new recursive schemes in the scope of the unified framework of section 2. A more comprehensive evaluation of these algorithms will be presented in a forthcoming paper.

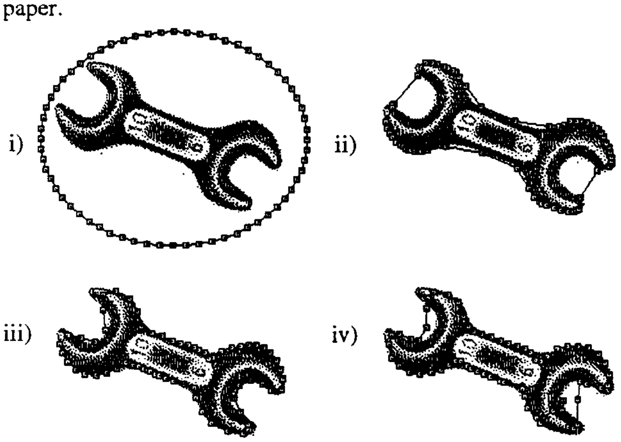

Figure 2 - Edge linking with deformable models
i) initial contour configuration; final contour shape with ii) snakes; iii) elastic nets and iv) Kohonen maps

6. Conclusion

This paper has presented an unified framework for edge linking operations which includes snakes, elastic nets and Kohonen maps as special cases. To design a new algorithm belonging to this framework all we have to do is to define a regularization matrix and a set of weighting functions which measure the influence of a pattern on each of the model units. The choice of the weighting functions is equivalent to the specification of the attraction regions associated with each model unit which have a simple geometrical interpretation. A detailed study of the experimental performance of snakes, elastic nets and Kohonen maps will be postponed to a forthcoming paper.

Finally, we note that the proposed framework is not restricted to edge linking operations and can be used to study other Pattern Recognition problems.

Appendix A - Proof of Fact 1

The complex gradient of (13) with respect to z_m is

$$\nabla_{z_m} J = \frac{d}{dz_m}\left(z^H A z + \sum_p \sum_{k=1}^{M} w_k(p) d_k(p) \right) \qquad (A1)$$

Using hypothesis (14, 15) the derivative of the inner sum is

$$\frac{d}{dz_m} \sum_k w_k(p) d_k(p) =$$

$$= \sum_k d_k(p) \frac{d}{dz_m} w_k(p) + \sum_k w_k(p) \frac{d}{dz_m} d_k(p) \qquad (A2)$$

$$= \sum_k d_k(p) \sum_j \frac{\partial w_k(p)}{\partial d_j(p)} \frac{\partial d_j(p)}{\partial z_m} + 2(z_m - p) w_m(p) \qquad (A3)$$

$$= 2(z_m - p)\left(\sum_k d_k(p) \frac{\partial w_k(p)}{\partial d_m(p)} + w_m(p) \right) \qquad (A4)$$

Replacing (A4) in (A1) we obtain (16, 17).

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