CALCULATION OF THE SAMPLE SELECTION PROBABILITIES OF STACK FILTERS BY USING WEIGHTED CHOW PARAMETERS

Pauli Kuosmanen, Karen Egiazarian and Jaakko Astola

Signal Processing Laboratory, Tampere University of Technology,
P.O.Box 553, SF-33101 Tampere, Finland
e-mail: pqo@cs.tut.fi, karen@cs.tut.fi, jta@cs.tut.fi

ABSTRACT

In the present work weighted Chow parameters are developed with the aim of their application in the statistical analysis of a class of nonlinear filters, namely stack filters, which are specified by positive Boolean functions (PBF) representing the binary output at each threshold level of the continuous-valued signal. Selection probabilities of stack filters were defined based on the fact that the output of a continuous stack filter is one of the samples within the input window. The notion of weighted Chow parameters is introduced in this paper for analysis and computation of the sample selection probability vector of a continuous stack filter.

1. INTRODUCTION

Notion of Chow parameters of a linearly separable Boolean function (BF) was introduced by C.K.Chow [2] with the aim of analysis of linearly separable Boolean functions (LSBF). Chow parameters have the property, that each LSBF has one unique non-zero set of Chow parameters, and no other function possesses this same set of values. So, these parameters are appropriate also for cataloguing of LSBFs [12].

Linearly separable Boolean functions form a subclass of the class of positive (monotonic) Boolean functions.

In a similar way class of nonlinear weighted order statistic filters (specified by LSBF) is a subclass of stack filters (specified by positive Boolean function at each threshold level of an input signal within the window) [1], [11], [13].

The concepts of selection probabilities for stack filters were introduced in [6], [9]. The sample selection probabilities give information about temporal-order information of stack filters.

In this paper the concepts of weighted activity of variables of the PBF and weighted Chow parameters are introduced for analysis of selection probabilities of stack filters. Based on this analysis we propose efficient algorithms to calculate sample selection probabilities of the stack filters.

2. SELECTION PROBABILITIES OF STACK FILTERS

2.1. Continuous Stack filters

Let $X(i)$, $i = 1, 2, ..., L$ be a continuous-valued input signal from the interval $[0,1]$, i.e. $0 \leq X(i) \leq 1$, and let a window of width $N = M + Q + 1$ slide accross the signal. At time instant n we have the following vector

$$\mathbf{X}(n) = [X(n-Q), ..., X(n), ..., X(n+M)] = [X_1(n), X_2(n), ..., X_N(n)] \quad (1)$$

within the window, where $X_j(n) = X(n+j-Q-1)$, $j = 1, 2, ..., N$.

Define the following positive *Continuous Logic Function (CLF)* (by analogue with positive Boolean functions, we call positive continuous logic function a continuous logic function which contains no complements of input variables)

$$F(X) = \max\big(\min(\Pi_1), \min(\Pi_2), ..., \min(\Pi_k)\big), \quad (2)$$

where

$$\min(\Pi_p) = \min(X_{j(p,1)}, X_{j(p,2)}, ..., X_{j(p,r_p)}), \quad (3)$$

$j(p,q)$ are the indices of the variables in Π_p in the increasing order and r_p is the number of variables in Π_p.

Definition 1. The output $Y(n)$ of a *continuous stack filter (CSF)* with the input signal $\mathbf{X}(n)$ specified by the positive continuous logic function $F(X)$ by

$$Y(n) = S_F(\mathbf{X}(n)) = F(\mathbf{X}(n)). \quad (4)$$

The continuous stack filter can also be defined [10], [12] by

$$Y(n) = S_F(\mathbf{X}(n)) = \max\{\beta | f(\sigma(\mathbf{X}(n), \beta)) = 1\}, \quad (5)$$

where $f(x)$ is a positive Boolean function, obtained from positive CLF $F(X)$ by changing the operations of max, min on the binary $\vee$, $\wedge$, respectively, $\sigma(\mathbf{X}(n), \beta) = (\sigma(X_1(n), \beta)$, $\sigma(X_2(n), \beta), ..., \sigma(X_N(n), \beta)$, and $\sigma(X_j(n), \beta)$ is a following hybrid *threshold operator*:

$$\sigma(X_j(n), \beta) = \begin{cases} 1, & \text{if } X_j(n) \geq \beta, \\ 0, & \text{otherwise.} \end{cases} \quad j = 1, 2, ..., N. \quad (6)$$

Formula (5) can be considered as one realization of the positive continuous logic function $F(X)$ by means of corresponding positive Boolean function $f(x)$ and hybrid threshold operator $\sigma(X, \beta)$.

The domain of definition of CLF is a set of points in the N-dimensional unit cube $[0,1]^N$, i.e. the set of points

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Region	Ordering	$F(X_1, X_2, X_3)$
1	$X_1 \le X_2 \le X_3$	X_1
2	$X_1 \le X_3 \le X_2$	X_1
3	$X_2 \le X_1 \le X_3$	X_1
4	$X_3 \le X_1 \le X_2$	X_1
5	$X_2 \le X_3 \le X_1$	X_3
6	$X_3 \le X_2 \le X_1$	X_2

Table 1: Table for CLF $F(X) = \max(\min(X_1, X_2), \min(X_1, X_3))$

Region k	Ordering	$Y = X_j = X_{(i)}$
1	$X_1 \le X_2 \le X_3$	$X_1 = X_{(1)}$
2	$X_1 \le X_3 \le X_2$	$X_1 = X_{(1)}$
3	$X_2 \le X_1 \le X_3$	$X_1 = X_{(2)}$
4	$X_3 \le X_1 \le X_2$	$X_1 = X_{(2)}$
5	$X_2 \le X_3 \le X_1$	$X_3 = X_{(2)}$
6	$X_3 \le X_2 \le X_1$	$X_2 = X_{(2)}$

Table 2: The Outputs of a Continuous Stack Filter Defined by PBF $Y = f(\mathbf{x}) = x_1x_2 \vee x_1x_3$

$(X_1, X_2, ..., X_N)$ of N-dimensional space $\mathbf{R}^N$, satisfying the conditions $0 \le X_j \le 1$, $j = 1, 2, ..., N$. The input variables $X_1, X_2, ..., X_N$ which we assume to be i.i.d. random variables may be permuted in $N!$ possible ways where each permutation is called an *ordering*. Any ordering z_k, $k = 1, 2, ..., N!$ is an arrangement of variables $X_1, X_2, ..., X_N$ such that $X_{k(1)} \le X_{k(2)} \le \ldots \le X_{k(N)}$, where $k(1), k(2), ..., k(N)$ is certain permutation of the numbers $1, 2, ..., N$. Thus, $[0,1]^N$ divides into $N!$ regions, each of which corresponds to a certain ordering of variables. Because CLF use only max and min operations in its definition, then on each point of the fixed region in $[0,1]^N$ the value of CLF coincides with the value of one variable (fixed for this region). Therefore, CLF can be given by the table where each regions associated to one of the variables. Consider the following example.

Example 1. Let the CLF $F(X) = \max(\min(X_1, X_2), \min(X_1, X_3))$ In the Table 1 are shown all possible $N! = 3! = 6$ orderings of the variables and corresponding values of the CLF.

2.2. Sample Selection Probabilities

Continuous stack filter can be given by the table, where any ordering $\mathcal{O}_k$ (from $N!$ possible orderings, $k = 1, 2, ..., N!$) is associated with the corresponding output of the continuous stack filter $Y = X_j$, $j = 1, 2, ..., N$. For any given ordering let the output Y be also the i^{th} ranked (smallest) input $X_{(i)}$, i.e. $Y = X_j = X_{(i)}$, $i, j = 1, 2, ..., N$.

In Table 2 the outputs of the continuous stack filter defined by the CLF $F(\bar{x}) = \max(\min(X_1, X_2), \min(X_1, X_3))$ for all possible orderings $\mathcal{O}_k$, $k = 1, 2, ..., 6$, are shown.

Sample selection probabilities are defined by Mallows [6]:

Definition 2 The j^{th} *Sample Selection Probability* is denoted by $P[Y = X_j]$, $1 \le j \le N$, and is the probability that the output $Y = X_j$. The *Sample Selection Probability Vector* is the row vector $\mathbf{s} = (s_1, s_2, ..., s_N)$, where $s_i = P[Y = X_j]$, $1 \le j \le N$.

Sample selection probability vector is given by

$$s_i = P[Y = X_j] = \sum_{i=1}^{N} \frac{(i-1)!(N-i)!C_{ij}}{N!}, \quad i = 1, 2, ..., N. \tag{7}$$

where C_{ij} is the number of distinct sets G_{ij}^k and H_{ij}^k, defined by, [8]:

$$G_{ij}^k = \{X_{(n)} \mid (n = 1, ..., i-1), Y = X_{(i)} = X_j\},$$

and

$$H_{ij}^k = \{X_{(n)} \mid (n = i+1, ..., N), Y = X_{(i)} = X_j\},$$

for the ordering $\mathcal{O}_k$.

As it is easy to see from Table 2 the sample selection probability vector of a stack filter based on the positive Boolean function $f(\mathbf{x}) = x_1x_2 \vee x_1x_3$ is equal to $\mathbf{r} = 1/6 \times [4\ 1\ 1]$.

3. CHOW PARAMETERS AND WEIGHTED CHOW PARAMETERS OF A BOOLEAN FUNCTION

The basic Chow parameters of a Boolean function $f(\mathbf{x})$ are

Definition 3 The vector of nonnegative integers

$$\mathbf{c} = (c_1, c_2, ..., c_N, c_0), \tag{8}$$

where

$$c_i = \left|\{\mathbf{x} \in \{0,1\}^N : f(\mathbf{x}) = 1,\ x_i = 1\}\right|, \quad (i = 1, 2, ..., N), \tag{9}$$

$$c_0 = |f^{-1}(1)|, \tag{10}$$

$$f^{-1}(1) = \{\mathbf{x} \in \{0,1\}^N : f(\mathbf{x}) = 1\}, \tag{11}$$

is called the *vector of Chow parameters of the Boolean function* $f(\mathbf{x})$.

In other words, c_i, $(i = 1, 2, ..., N)$ is the sum of the occurrences of x_i taken over all true vectors, and c_0 is the total number of true vectors.

In Table 3 the Chow parameters for the PBF $f(\mathbf{x}) = x_1x_2 \vee x_1x_3$ are shown.

As we see, each value c_j, $j = 1, 2, ..., N$, corresponds to the number of times each variable x_j occurs in the minterm expansion, ignoring the presence of $\bar{x}_j$.

Now we generalize the concept of Chow parameters.

Definition 4 Let $k \in \{0, 1\}$. The vector $\mathbf{d}(k)$ of *weighted Chow parameters* of a Boolean function $f(\mathbf{x})$ of n variables is given by

$$\mathbf{d}(k) = (d_1(k), d_2(k), ..., d_n(k)), \tag{12}$$

	True Minterms				
1	$x_1x_2\bar{x}_3$	1	1	0	
2	$x_1x_2x_3$	1	1	1	
3	$x_1\bar{x}_2x_3$	1	0	1	
Total	c	3	2	2	3

Table 3: Chow Parameters for a Positive Boolean Function $f(\mathbf{x}) = x_1x_2 \vee x_1x_3$

	True Minterms						
1	$x_1x_2\bar{x}_3$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0	$\frac{1}{3}$
2	$x_1x_2x_3$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0
3	$x_1\bar{x}_2x_3$	$\frac{1}{6}$	0	$\frac{1}{6}$	0	$\frac{1}{3}$	0
Total	d(1)	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$			
Total	d(0)				0	$\frac{1}{3}$	$\frac{1}{3}$

Table 4: Weighted Chow Parameters for a Positive Boolean Function $f(\mathbf{x}) = x_1x_2 \vee x_1x_3$

where

$$d_i(k) = \sum_{\alpha \in f^{-1}(1) | \alpha_i = k} \pi(\alpha^{(i)}), \tag{13}$$

and $\pi(\alpha^{(i)})$ are defined by

$$\pi(\alpha^{(i)}) = \frac{1}{N \cdot \binom{N-1}{w_H(\alpha^{(i)})}}, \tag{14}$$

$$\alpha^{(i)} = (\alpha_1, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_N), \tag{15}$$

$(i = 1, 2, ..., N)$, $w_H(\alpha^{(i)})$ is the Hamming weight of the vector $\alpha^{(i)}$, i.e. the number of ones in $\alpha^{(i)}$, and $f^{-1}(1)$ is defined by (11).

In Table 4 the weighted Chow parameters are shown for the same positive Boolean function as in the Table 3.

4. SAMPLE SELECTION PROBABILITY VECTORS AND WEIGHTED CHOW PARAMETERS

Sample selection probabilities can be calculated by (7) using the coefficients C_{ij}. It is known, (see [3], [5]) that

$$C_{ij} = \sum_{\alpha | w_H(\alpha^{(j)}) = N-i} \frac{\partial f(\alpha)}{\partial x_j}, \tag{16}$$

where $\alpha^{(j)}$ is defined by (15), $\frac{\partial f(\alpha)}{\partial x_j} = f(\alpha^{(j,1)}) \oplus f(\alpha^{(j,0)})$ $= f(\alpha_1, ..., \alpha_{j-1}, 1, \alpha_{j+1}, ..., \alpha_N) \oplus f(\alpha_1, ..., \alpha_{j-1}, 0, \alpha_{j+1}, ..., \alpha_N)$ is the partial derivative of $f(\alpha)$ with respect of variable x_j, [9].

The following proposition holds.

Proposition 1. The sample selection probability vector s of the continuous stack filter based on the PBF $f(\mathbf{x})$ of N variables can be constructed using vectors of weighted Chow parameters d(1) and d(0) of $f(\mathbf{x})$ by the following expression:

$$\mathbf{s} = \mathbf{d}(1) - \mathbf{d}(0). \tag{17}$$

Proof. From the equations (7), (16) and from the definition of partial derivative we have

$$\mathbf{s} = \sum_{\alpha^{(j)} \in \{0,1\}^{N-1}} \pi(\alpha^{(j)}) \cdot \frac{\partial f(\alpha^{(j)})}{\partial x_j} = \sum_{\alpha^{(j)} \in \{0,1\}^{N-1}} \pi(\alpha^{(j)}) \cdot [f(\alpha^{(j,1)}) \oplus f(\alpha^{(j,0)})] = \sum_{\alpha \in \{0,1\}^N} \pi(\alpha) \cdot |f(\alpha^{(j,1)}) - f(\alpha^{(j,0)})|$$

Because $f(\mathbf{x})$ is a positive Boolean function, i.e., $f(\alpha^{(j,1)}) \geq f(\alpha^{(j,0)})$ for all $j = 1, 2, ..., N$, we have

$$\mathbf{s} = \sum_{\alpha^{(j)} \in \{0,1\}^{N-1}} \pi(\alpha^{(j)}) \cdot f(\alpha^{(j,1)}) - - \sum_{\alpha^{(j)} \in \{0,1\}^{N-1}} \pi(\alpha^{(j)}) \cdot f(\alpha^{(j,0)}) = = \sum_{\alpha \in f^{-1}(1) | \alpha_j = 1} \pi(\alpha^{(j)}) - \sum_{\alpha \in f^{-1}(1) | \alpha_j = 0} \pi(\alpha^{(j)}) = = \mathbf{d}(1) - \mathbf{d}(0).$$

Thus, we have proved this proposition. □

True Minterms Π_k	Codes of $\alpha \in f^{-1}(1)$	$\pi(\alpha^j)$ if $\alpha_j = 1$ or $-\pi(\alpha^j)$ if $\alpha_j = 0,\ j = 1, 2, 3, 4$			
$x_1x_2\bar{x}_3\bar{x}_4$	1100	$\frac{1}{12}$	$\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{12}$
$x_1x_2\bar{x}_3x_4$	1101	$\frac{1}{12}$	$\frac{1}{12}$	$-\frac{1}{4}$	$\frac{1}{12}$
$x_1x_2x_3\bar{x}_4$	1110	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$-\frac{1}{4}$
$x_1x_2x_3x_4$	1111	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$\bar{x}_1x_2x_3\bar{x}_4$	0110	$-\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$-\frac{1}{12}$
$\bar{x}_1x_2x_3x_4$	0111	$-\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
$x_1\bar{x}_2x_3x_4$	1011	$\frac{1}{12}$	$-\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$
Total	s	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{1}{12}$

Table 5: Sample Selection Probability Vector s for a Continuous Stack Filter Defined by PBF $f(\mathbf{x}) = x_1x_2 \vee x_2x_3 \vee x_1x_3x_4$

In Table 5 we show an example of finding the sample selection probability vector s of the continuous stack filter based on the 4 variable PBF $f(\mathbf{x}) = x_1x_2 \vee x_2x_3 \vee x_1x_3x_4$.

Proposition 2 The sample selection probability vector s of the continuous stack filter based on the N variable PBF $f(\mathbf{x})$ can be constructed using vectors of weighted Chow parameters of $f^D(\mathbf{x})$ (the dual of the PBF $f(\mathbf{x})$) by the following expression:

$$\mathbf{s} = \mathbf{d}^{f^D}(1) - \mathbf{d}^{f^D}(0), \tag{18}$$

where $\mathbf{d}^{f^D}(k)$ is defined for PBF $f^D(\mathbf{x})$ as $\mathbf{d}(k)$ is for $f(\mathbf{x})$, $k \in \{0, 1\}$, i.e., $\mathbf{d}^{f^D}(k) = (d_1^{f^D}, d_2^{f^D}, ..., d_N^{f^D})$, with

$$d_i^{f^D}(k) = \sum_{\alpha \in (f^D)^{-1}(1) | \alpha_j = k} \pi(\alpha^{(j)}),\ k \in \{0, 1\}. \tag{19}$$

Proof is the same as the proof of Proposition 1.

True Minterms Π_k of dual PBF f^D	Codes of $\alpha \in (f^D)^{-1}(1)$ of dual PBF	$\pi(\alpha^j)$ if $\alpha_j = 1$ or $-\pi(\alpha^j)$ if $\alpha_j = 0$, $j = 1, 2, 3, 4$
$\bar{x}_1 x_2 x_3 \bar{x}_4$	0110	$-\frac{1}{12}\ \frac{1}{12}\ \frac{1}{12}\ -\frac{1}{12}$
$\bar{x}_1 x_2 \bar{x}_3 x_4$	0101	$-\frac{1}{12}\ \frac{1}{12}\ -\frac{1}{4}\ \frac{1}{12}$
$\bar{x}_1 x_2 x_3 x_4$	0111	$-\frac{1}{4}\ \frac{1}{12}\ \frac{1}{12}\ \frac{1}{12}$
$x_1 \bar{x}_2 x_3 \bar{x}_4$	1010	$\frac{1}{12}\ -\frac{1}{12}\ \frac{1}{12}\ -\frac{1}{12}$
$x_1 \bar{x}_2 x_3 x_4$	1011	$\frac{1}{12}\ -\frac{1}{4}\ \frac{1}{12}\ \frac{1}{12}$
$x_1 x_2 \bar{x}_3 \bar{x}_4$	1100	$\frac{1}{12}\ \frac{1}{12}\ -\frac{1}{12}\ -\frac{1}{12}$
$x_1 x_2 \bar{x}_3 x_4$	1101	$\frac{1}{12}\ \frac{1}{12}\ -\frac{1}{4}\ \frac{1}{12}$
$x_1 x_2 x_3 \bar{x}_4$	1110	$\frac{1}{12}\ \frac{1}{12}\ \frac{1}{12}\ -\frac{1}{4}$
$x_1 x_2 x_3 x_4$	1111	$\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}$
Total	s	$\frac{1}{4}\ \frac{5}{12}\ \frac{1}{4}\ \frac{1}{12}$

Table 6: Sample Selection Probability Vector s for a Continuous Stack Filter Defined by PBF $f(\mathbf{x}) = x_1x_2 \vee x_2x_3 \vee x_1x_3x_4$ Using Dual PBF $f^D(\mathbf{x}) = x_1x_3 \vee x_1x_2 \vee x_2x_3 \vee x_2x_4$

In Table 6 we show another example of finding of the sample selection probability vector s of the continuous stack filter based on the PBF $f(\mathbf{x}) = x_1x_2 \vee x_2x_3 \vee x_1x_3x_4$, by the corresponding dual PBF $f^D(\mathbf{x}) = x_1x_3 \vee x_1x_2 \vee x_2x_3 \vee x_2x_4$. The result is the same as in the Table 5.

Note that recalculation of the Chow-parameters in the classification of all linearly separable functions of $N \leq 7$ variables (given by Winder, [11]) on the terms of weighted Chow-parameters give us possibility to solve two problems simultaneously:

1. Find out if a given PBF is linearly separable.
2. Obtain the sample selection probability vector $\mathbf{s} = (s_1, s_2 ..., s_N)$ (with the possibility of further closely approximation of stack (nonlinear) filter based on this PBF by a FIR (linear) filter with an impulse response $\mathbf{h} = (s_N, ..., s_2, s_1)$, [6]).

5. CONCLUSION

In this work we consider the problem of computation of sample selection probabilities for stack filters.

A new notation of weighted Chow parameters for any Boolean function is presented. The connection between the vectors of weighted Chow parameters and the sample selection probability vector s of a stack filter is established. Two possibilities of calculation of the vector s are presented: by the Boolean function itself and by its dual.

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