DIGITAL ESTIMATION OF FREQUENCIES OF SINUSOIDS FROM WIDE-BAND UNDER-SAMPLED DATA

Donald W. Tufts and Hongya Ge

Department of Electrical Engineering, The University of Rhode Island, Kingston RI 02881

ABSTRACT

We present an novel method of fast and accurate estimation of frequencies of sinusoids from short data records of wide-band under-sampled data. By introducing properly chosen delay lines, and by using sparse linear prediction [1, 2, 3], our proposed method provides unambiguous frequency estimates using low A/D conversion rates. It provides a new way to implement a digital microwave receiver under these conditions.

1. INTRODUCTION

In this work, we present an novel method of fast and accurate estimation of frequencies of sinusoids from short data records of wide-band under-sampled data. This is an extension of our previous results on Sparse Linear Prediction (SLP) [1, 2] and on the design a sparse array of electromagnetic or acoustic sensors [3]. Such a digital receiver implementation can be useful for wide-band high-frequency applications in wireless communications and signal analysis systems. In these applications, A/D conversion and digital signal processing hardware and software are not sufficiently fast for Nyquist rate processing. A significant part of this work is that it provides a way to implement a digital receiver under these conditions. Related to our work in its objective is the paper of Rader [4] on the recovery of periodic waveforms from undersampled data. Our work is mainly concentrated on estimating frequencies of sinusoids.

2. PROPOSED ESTIMATION APPROACH

Let $y(t)$ be a continuous-time waveform consisting of a linear combination of M exponentials with imaginary arguments. Figure 1 below shows the configuration of the system we propose to estimate the frequencies of the sinusoids from the under-sampled data. The sampling period t_s might not satisfy the Nyquist sampling theorem. The main steps of the algorithm are summarize as follows,

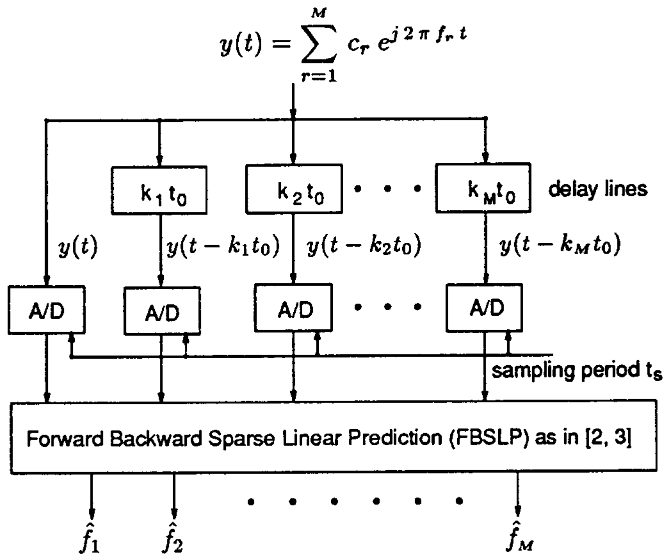

Figure 1. The configuration of the proposed digital receiver

Step 1. Choosing delays $k_1 t_0,\ k_2 t_0,\ \ldots,\ k_M t_0$, such that k_i's are relatively prime. The value of t_0 is chosen so that $(0,\ 1/t_0)$ corresponds to the desired unambiguous frequency band, in which the incoming waveform $y(t)$ is assumed to lie, or $t_0 < \min\left\{\frac{1}{f_1}, \ldots, \frac{1}{f_M}\right\}$.

Step 2. Sampling $y(t)$ and all its delayed versions $y(t-k_1 t_0), \ldots, y(t-k_M t_0)$ with sampling period t_s to form the FBSLP data matrix $\mathbf{Y}$.

Step 3. Solving for vector $\hat{\mathbf{a}}$ from the linear prediction equation $\mathbf{Y} \cdot \begin{bmatrix} 1 \\ -\hat{\mathbf{a}} \end{bmatrix} = 0$.

Step 4. Evaluating angles of the unit circle roots of the KLP polynomial $G(z) = 1 - \sum_{i=1}^{M} a_{k_i} z^{-k_i}$ to get the estimates of frequencies.

Note that the sampling period t_s and the delays $k_1 t_0$, $k_2 t_0,\ \ldots,\ k_M t_0$ should be chosen such that the rank of matrix $\mathbf{Y}$ is at least M. Then $\hat{\mathbf{a}}$ can be solved from a $M \times M$ matrix equation uniquely in least square sense.

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The formation of matrix $\mathbf{Y}$ reveals the rationale of using SLP to estimate frequencies from the under-sampled wide-band data. Specifically, the matrix $\mathbf{Y}$ is formed in the following way,

$$\mathbf{Y} = [\mathbf{y}_0 \ \mathbf{y}_1 \ \cdots \ \mathbf{y}_M] \tag{1}$$

with

$$\mathbf{y}_0 = \begin{bmatrix} y[(L+1)\,t_s] \\ y[(L+2)\,t_s] \\ \vdots \\ y[N\,t_s] \\ y^*[t_s] \\ y^*[2\,t_s] \\ \vdots \\ y^*[(N-L)\,t_s] \end{bmatrix}, \mathbf{y}_i = \begin{bmatrix} y[(L+1)\,t_s - k_i\,t_0] \\ y[(L+2)\,t_s - k_i\,t_0] \\ \vdots \\ y[N\,t_s - k_i\,t_0] \\ y^*[t_s + k_i\,t_0] \\ y^*[2\,t_s + k_i\,t_0] \\ \vdots \\ y^*[(N-L)\,t_s + k_i\,t_0] \end{bmatrix},$$

$$i = 1, 2, \ldots, M.$$

In other words, columns of $\mathbf{Y}$ are formed by sampling $y(t)$, and its delayed versions $y(t-k_1\,t_0), \ldots, y(t-k_M\,t_0)$ with sampling period t_s. In general $t_s \geq t_0$ holds, even though these time series data are not sampled at high enough Nyquist rate $1/t_0$, the properly chosen delays between columns of matrix $\mathbf{Y}$ can help to extract the information about the actual frequencies of the sinusoids in the time series data. Inspired by the ideas in array signal processing, we can always decompose matrix $\mathbf{Y}$ as follows,

$$\mathbf{Y} = \mathbf{D} \cdot \mathbf{M} \tag{2}$$

where columns of $\mathbf{D}$ matrix contain the sub-Nyquist samples of each frequency component of the data; and rows of $\mathbf{M}$ matrix contain the exponentials which indicate the phase shift of each frequency component due to the use of delay lines. When the values of delays and sampling period are chosen such that the matrix $\mathbf{D}$ is of full rank, the frequencies can be estimated by first solving a from the following FBSLP equation

$$\mathbf{Y} \cdot \begin{bmatrix} 1 \\ -\mathbf{a} \end{bmatrix} = \mathbf{0}, \quad \text{with} \quad \mathbf{a} = [a_{k_1}\, a_{k_2} \cdots a_{k_M}]^T \tag{3}$$

then rooting the KLP polynomial

$$G(z) = 1 - \sum_{i=1}^{M} a_{k_i}\, z^{-k_i} \tag{4}$$

The only possible ambiguity is caused by the use of large delays. Therefore, the method of resolving ambiguities we used in [2] can also be used here for resolving the ambiguities due to the large delays. As proposed in [2], in order to resolve this ambiguity, an additional KLP polynomial should be formed by solving another set of FBSLP equations, with another set of delays $l_1\,t_0,\ l_2\,t_0,\ \ldots,\ l_M\,t_0$ (see Figure 1). The criterion to choose delays proposed in [2] could also be used for this generalized case.

2.1. SINGLE SINUSOID CASE

In the case of single sinusoid, we assume that two different values of delays $\tau_1 = k_1\,t_0$ and $\tau_1' = l_1\,t_0$ are used. The delay integers k_1 and l_1 are chosen to be relatively prime, or $(k_1, l_1) = 1$. The continuous-time waveform $y(t)$ and its delayed versions $y(t-\tau_1)$, $y(t-\tau_1')$ are sampled at sub-Nyquist rate $1/t_s$, where $t_s \geq t_0$ holds in general. Using waveforms $y(t)$ and its delayed version $y(t-\tau_1)$, the data matrix $\mathbf{Y}$, which is used in the sparse linear prediction, can then be decomposed as follows,

$$\begin{aligned} \mathbf{Y} &= \begin{bmatrix} y(t_s) & y(t_s - \tau_1) \\ y(2\,t_s) & y(2\,t_s - \tau_1) \\ \vdots & \vdots \\ y(N\,t_s) & y(N\,t_s - \tau_1) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} c_1\, e^{j\,2\,\pi\,f_1\,t_s} \\ c_1\, e^{j\,2\,\pi\,f_1\,2\,t_s} \\ \vdots \\ c_1\, e^{j\,2\,\pi\,f_1\,N\,t_s} \end{bmatrix}}_{\mathbf{D}} \cdot \underbrace{\begin{bmatrix} 1 & e^{-j\,2\,\pi\,f_1\,\tau_1} \end{bmatrix}}_{\mathbf{M}_1} \end{aligned} \tag{5}$$

Note that the matrix $\mathbf{D}$ is always full rank. Therefore, we have the following equivalent equations,

$$\mathbf{Y} \cdot \begin{bmatrix} 1 \\ -a_{k_1} \end{bmatrix} = 0 \quad \longleftrightarrow \quad \mathbf{M}_1 \cdot \begin{bmatrix} 1 \\ -a_{k_1} \end{bmatrix} = 0 \tag{6}$$

In other words, since $\tau_1 = k_1\,t_0$, the frequency of the sinusoid can be estimated by evaluating the unit circle zero of the KLP polynomial defined by

$$G_1(z) = 1 - a_{k_1}\, z^{-k_1} \tag{7}$$

where a_{k_1} is solved from (6) in the least square sense.

As mentioned in [2, 5], in order to reduce the noise sensitivity of the frequency estimator, large value of delay integer k_1 is chosen. This will cause ambiguity in determining the frequency from the unit circle zero of the KLP polynomial $G_1(z)$. Since there will be k_1 unit circle zeros, among which the true signal zero lies. Following the same procedures of resolving ambiguity proposed in [2], we can use $y(t)$ and the other delayed waveform $y(t-\tau_1') = y(t - l_1\,t_0)$ to form a second set of sparse linear prediction as (6),

$$\mathbf{Y} \cdot \begin{bmatrix} 1 \\ -a_{l_1} \end{bmatrix} = 0 \quad \longleftrightarrow \quad \mathbf{M}_2 \cdot \begin{bmatrix} 1 \\ -a_{l_1} \end{bmatrix} = 0 \tag{8}$$

with $\mathbf{M}_2 = \left[\begin{array}{cc} 1 & e^{-j2\pi f_1 \tau_1'} \end{array} \right]$. Then we can get a second KLP polynomial after solving for a_{l_1} from (8),

$$G_2(z) = 1 - a_{l_1}\, z^{-l_1} \tag{9}$$

Since the values of the delay integers k_1 and l_1 are chosen to be relatively prime, the true signal zero is given by the only common unit circle zeros of both KLP polynomials $G_1(z)$ in (7) and $G_2(z)$ in (9).

When noise is present, we can form an equivalent objective function by combining both KLP polynomials as follows,

$$P(f) = \frac{1}{|\,G_1(e^{j2\pi f t_0})\,|^2 + |\,G_2(e^{j2\pi f t_0})\,|^2} \tag{10}$$

Then the frequency of the sinusoid can be estimated by search for the global maximum of the above $P(f)$.

In practice, when the available data record is finite, there exist limitations on the choice of sampling period t_s, delays $k_1 t_0$ and $l_1 t_0$. Small values of sampling period t_s can help to get more LP equations, but it is limited by hardware and software speed. Large values of delays $k_1 t_0$ and $l_1 t_0$ can help to get better frequency estimates [2, 5], but they are limited by the available data record length. We will provide detailed simulation results in the later sections.

2.2. TWO SINUSOID CASE

When the received waveform consists two sinusoid components, we need an additional delayed waveform to get different phase shifts, which are necessary to extract the frequency information. Choosing one set of delays $\tau_1 = k_1 t_0$ and $\tau_2 = k_2 t_0$, we sample the continuous-time waveform $y(t) = c_1 e^{j2\pi f_1 t} + c_2 e^{j2\pi f_2 t}$ and its delayed versions $y(t-\tau_1)$ and $y(t-\tau_2)$ at a rate $1/t_s$. We then decompose the data matrix $\mathbf{Y}$, which is needed to form the sparse linear prediction equations, as follows,

$$\begin{aligned} \mathbf{Y} &= \begin{bmatrix} y(t_s) & y(t_s - \tau_1) & y(t_s - \tau_2) \\ y(2\,t_s) & y(2\,t_s - \tau_1) & y(2\,t_s - \tau_2) \\ \vdots & \vdots & \vdots \\ y(N\,t_s) & y(N\,t_s - \tau_1) & y(N\,t_s - \tau_2) \end{bmatrix} \\ &= \mathbf{D} \cdot \mathbf{M}_1 \end{aligned} \tag{11}$$

with $$\mathbf{D} = \begin{bmatrix} c_1\, e^{j2\pi f_1 t_s} & c_2\, e^{j2\pi f_2 t_s} \\ c_1\, e^{j2\pi f_1 2 t_s} & c_2\, e^{j2\pi f_2 2 t_s} \\ \vdots & \vdots \\ c_1\, e^{j2\pi f_1 N t_s} & c_2\, e^{j2\pi f_2 N t_s} \end{bmatrix}$$

and $$\mathbf{M}_1 = \begin{bmatrix} 1 & e^{-j2\pi f_1 \tau_1} & e^{-j2\pi f_1 \tau_2} \\ 1 & e^{-j2\pi f_2 \tau_1} & e^{-j2\pi f_2 \tau_2} \end{bmatrix}$$

If the matrix $\mathbf{D}$ in (11) is full rank, the we have the following equivalent equations,

$$\mathbf{Y} \cdot \begin{bmatrix} 1 \\ -a_{k_1} \\ -a_{k_2} \end{bmatrix} = 0 \quad \longleftrightarrow \quad \mathbf{M}_1 \cdot \begin{bmatrix} 1 \\ -a_{k_1} \\ -a_{k_2} \end{bmatrix} = 0 \tag{12}$$

But due to the effect of under-sampling in matrix $\mathbf{D}$, the matrix $\mathbf{D}$ will become rank deficient whenever $|f_2 - f_1| = r/t_s$, where r is any integer. In this work, we solve this problem by using a swept sampling scheme, which will guarantee the full rank of the $\mathbf{D}$ matrix. Then the frequencies can be estimated by solving for $[a_{k_1}\ a_{k_2}]^T$ from (12) followed by rooting the KLP polynomial

$$G_1(z) = 1 - a_{k_1}\, z^{-k_1} - a_{k_2}\, z^{-k_2}\,. \tag{13}$$

Similarly, in order to resolve possible ambiguities due to the use of large delays and SLP, another set of delays $\tau_1' = l_1 t_0$ and $\tau_2' = l_2 t_0$ is chosen following the same criterion in [2]. After solving for SLP coefficient $[a_{l_1}\ a_{l_2}]^T$, an additional KLP polynomial is obtained as,

$$G_2(z) = 1 - a_{l_1}\, z^{-l_1} - a_{l_2}\, z^{-l_2}\,. \tag{14}$$

Then the frequencies can be unambiguously estimated from the only two common unit circle zeros of these two KLP polynomials. Or by combining both KLP polynomials as in (10), we can also unambiguously estimate the frequencies from the distinct peaks of the objective function $P(f)$ in (10).

3. SIMULATION RESULTS

Simulation results using the proposed FBSLP method to estimate frequencies of sinusoids from under-sampled data are shown in Figure 2 through Figure 4. In these examples, the sampling period is assumed to be $t_s = 11$, but the required Nyquist sampling rate is $1/t_0 = 1$. Figure 2 shows the results of a single sinusoid case. The results of two sinusoid case are shown in Figure 3 and Figure 4. Figure 3 shows the high resolution ability of proposed method in resolving two closely spaced sinusoids. Figure 4 shows its ability of resolve wide-band ambiguities in resolving two widely spaced sinusoids. All the plots are superimposed results of 50 independent trials. The true signal zeros are shown by arrows and true signal frequencies are shown by dashed lines.

Note that in all these cases, ambiguities might occur in individual KLP polynomial (see Figure (a)'s-(d)'s), When both KLP polynomials are combined together, no more ambiguity is observed in Figure (e)'s. The frequencies can be estimated from the co-ordinates of the distinct peaks of the objective function $P(f)$ in Figure (e)'s, which combines both KLP polynomials.

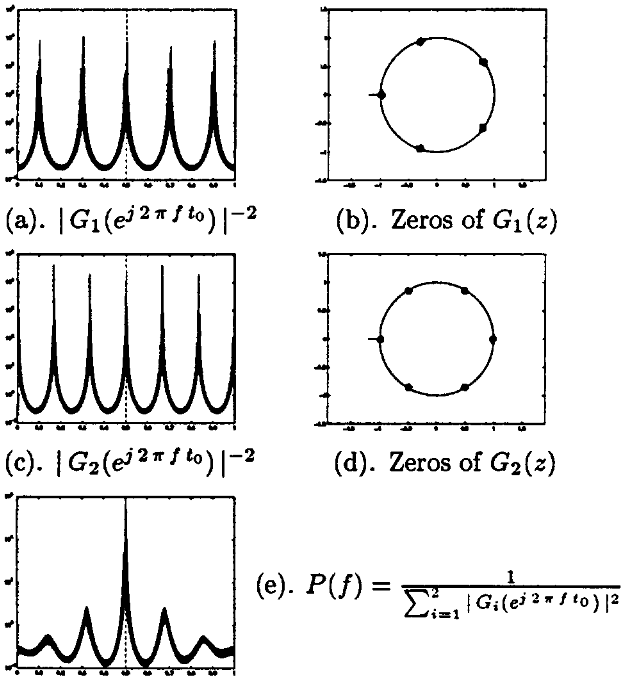

Figure 2. $P(f)$ and root positions of polynomials $G_1(z) = 1 - a_5\, z^{-5}$ and $G_2(z) = 1 - a_6\, z^{-6}$ in the case of a single sinusoid. Parameters: $N = 18$, $t_s = 11$, $t_0 = 1$, $SNR = 10\,db$, $\omega_1 = 2\pi\, 0.5$.

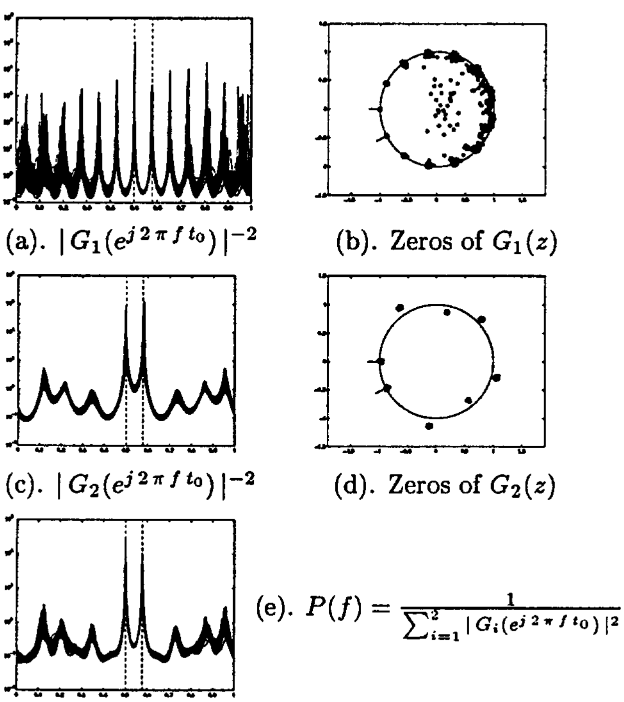

Figure 3. $P(f)$ and root positions of polynomials $G_1(z) = 1 - a_{13}\, z^{-13} - a_{14}\, z^{-14}$ and $G_2(z) = 1 - a_5\, z^{-5} - a_8\, z^{-8}$ in the case of two sinusoids. Parameters: $N = 18$, $t_s = 11$, $t_0 = 1$, $SNR = 10\,db$, $\omega_1 = 2\pi\, 0.5$, $\omega_2 = \omega_1 + 2\pi \frac{1}{13}$.

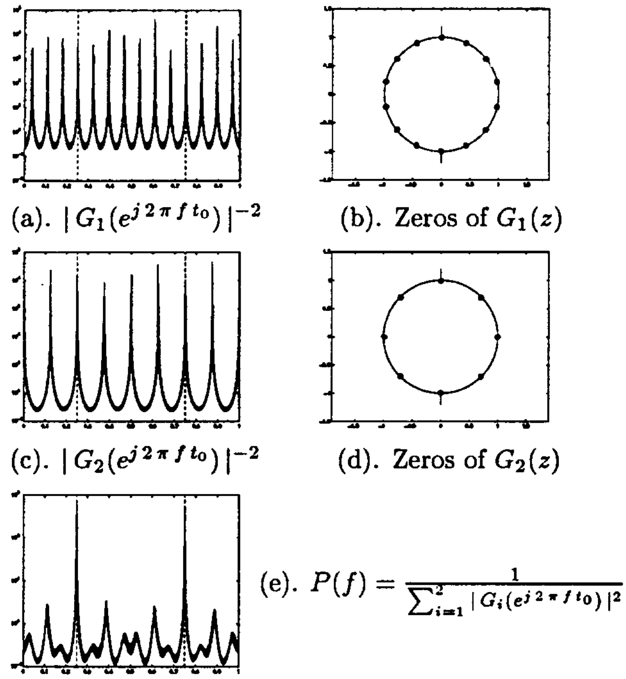

Figure 4. $P(f)$ and root positions of polynomials $G_1(z) = 1 - a_{13}\, z^{-13} - a_{14}\, z^{-14}$ and $G_2(z) = 1 - a_5\, z^{-5} - a_8\, z^{-8}$ in the case of two sinusoids. Parameters: $N = 18$, $t_s = 11$, $t_0 = 1$, $SNR = 10\,db$, $\omega_1 = 2\pi\, 0.25$, $\omega_2 = \omega_1 + \pi$.

4. CONCLUSION

A fast and accurate method of estimating frequencies of sinusoids from wide-band under-sampled data is provided. The importance of this work is that it significantly reduces the hardware/software cost in many applications such as cellular and mobile communications and RF electronics, where Nyquist sampling is almost impossible.

5. REFERENCES

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