# AN IMPROVEMENT TO THE EXPLICIT TIME DELAY ESTIMATOR

H.C. So*, P.C. Ching * and Y.T. Chan **

*Dept. of Electronic Engineering, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong
**Dept. of Electrical and Computer Engineering, Royal Military College of Canada
Kingston, Ontario, K7K, 5LO, Canada

## ABSTRACT

The explicit time delay estimator (ETDE) provides an efficient way to estimate the time difference of arrival between signals received at two separated sensors. However, the algorithm is biased for finite filter length and the delay bias increases when the signal-to-noise ratio (*SNR*) or the number of filter taps decreases. In this paper, we add an adaptive gain control to the ETDE to decouple the effect of changes in the *SNR* during adaptation. As a result, a smaller delay variance and an unbiased delay estimate for a wide range of filter lengths can be attained. Computer simulations are presented to validate the theoretical derivations of the proposed estimator for static and linearly time-varying delays under both stationary and nonstationary signal/noise power environments.

## I. INTRODUCTION

The estimation of the time delay between the outputs of two spatially separated sensors has found important applications in sonar, radar, global positioning system (GPS), biomedical engineering and so on [1-3]. For example, in GPS, the location of an interfering transmitter which tampers with satellite operations can be determined using the differential satellite path delay measurements [3].

The two received signals are

$$x(k) = s(k) + n_1(k) \tag{1a}$$

$$y(k) = s(k - D(k)) + n_2(k) \tag{1b}$$

where the source signal $s(k)$ and the corrupting noises $n_1(k)$ and $n_2(k)$ are Gaussian, stationary and mutually uncorrelated white processes. Without loss of generality, we assume that the signal and the noise spectra are band-limited between $-\pi$ and $\pi$ while the sampling period is unity. The signal power is represented by $\sigma_s^2$ while $\sigma_n^2$ denotes the power of $n_1(k)$ and $n_2(k)$. The objective is to estimate the time difference of arrival, $D(k)$, which may be time-varying, from $x(k)$ and $y(k)$.

Recently, a new adaptive structure called the explicit time delay estimator (ETDE) [4-6] has been proposed for the TDE problem. It has two obvious advantages over the conventional generalized cross correlation (GCC) methods [7,8]. Firstly, it does not require spectral estimation of the transmitted source nor the noise signal. Secondly, it can track nonstationary delays due to the relative motion between the source and the sensors. The ETDE is similar to the least mean square time delay estimator (LMSTDE) [9,10] in the sense that they both model the time delay by using an FIR filter. But unlike the LMSTDE where the estimated delay is obtained indirectly by interpolating the filter coefficients, the ETDE updates the delay estimate directly and involves no interpolation in the adaptation.

In this paper, we examine the performance of the ETDE rigourously, particularly its biasedness under low *SNR* conditions. It will be shown that the delay bias is a function of the *SNR*, the number of filter taps and the actual delay. By adding an adjustable gain control to the ETDE as shown in Figure 1, it is proved that the delay estimate of this proposed estimator is unbiased for all finite filter lengths and is independent of the *SNR*. We call this an explicit time delay and gain estimator (ETDGE). Theoretical analysis also shows that the improved algorithm will give smaller variances at low *SNR* for both static and nonstationary delays and this is confirmed via simulation results.

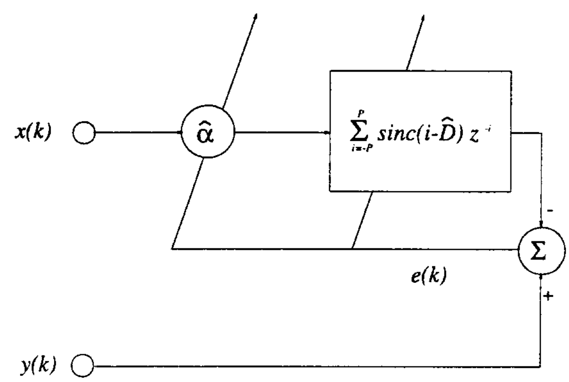


Figure 1 A generalized explicit time delay and gain estimator (ETDGE)

## II. BIASEDNESS OF THE ETDE

From Figure 1, if we fix $\hat{\alpha} = 1$, we obtain the configuration for the ETDE [4]. The filter coefficients of this estimator, $\{sinc(i - \hat{D}(k))\}$ for $-P \le i \le P$, are a function of the delay estimate, $\hat{D}(k)$, only. From the convolution theorem, this filter will provide an exact time lag of $\hat{D}(k)$ to $x(k)$ when infinite filter length is used. The delay modeling error due to the finite filter length, $2P+1$, has been discussed in [11] and it has been proved that the truncation error decreases as $P$ increases. The output error, $e(k)$, is given by

$$e(k) = y(k) - \sum_{i=-P}^{P} sinc(i - \hat{D}(k))\, x(k-i) \tag{2}$$



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Similar to Widrow's LMS algorithm, the ETDE uses a stochastic gradient estimate which is obtained by differentiating the instantaneous square error, $e^2(k)$, with respect to $\hat{D}(k)$. The estimated delay is updated at each iteration according to

$$\hat{D}(k+1) = \hat{D}(k) - \mu \frac{\partial e^2(k)}{\partial \hat{D}(k)}$$

$$= \hat{D}(k) - 2\mu e(k) \sum_{i=-P}^{P} x(k-i) f(i-\hat{D}(k)) \quad (3)$$

where $f(v) = (\cos(\pi v) - sinc(v))/v$ and $\mu$ is a parameter that controls convergence rate and stability of the algorithm.

We shall first analyse the performance of the ETDE assuming that $D(k)$ is a constant of value, say $D$. When $P$ is chosen sufficiently large, it has been shown in [4] that $\hat{D}(k)$ is unbiased as $k$ goes to infinity. However, in practice, a finite filter length is used and in this case the delay estimate is usually biased. It is well known that the mean delay estimate, which is denoted by $\overline{D}$, should correspond to the global minimum of the performance surface $E\{e^2(k)\}$ where $E$ represents the expectation operation. By differentiating $E\{e^2(k)\}$ obtained from (2) with respect to $\hat{D}$ and then equating the result to zero, we obtain the relationship between $\overline{D}$, $D$, the $SNR$ and $P$ as

$$\sum_{i=-P}^{P} sinc(i-D) f(i-\overline{D}) =$$

$$(1+SNR^{-1}) \sum_{i=-P}^{P} sinc(i-\overline{D}) f(i-\overline{D}) \quad (4)$$

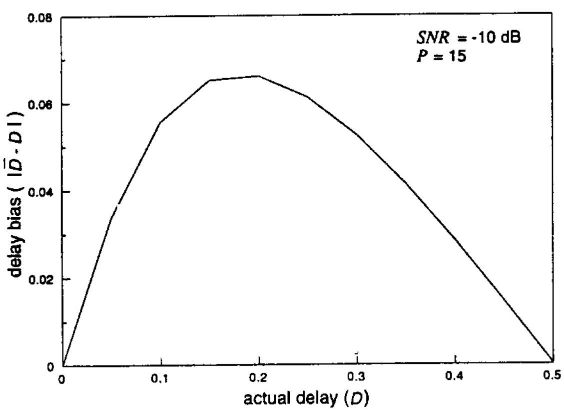


Figure 2 Delay bias versus $D$

| | | *SNR* (dB) | | | |
|---|---|---|---|---|---|
| | | -20 | 0 | 20 | 40 |
| *P* | 5 | $2.364\times10^{-1}$ | $2.101\times10^{-2}$ | $2.126\times10^{-4}$ | $2.126\times10^{-6}$ |
| | 50 | $1.327\times10^{-1}$ | $1.947\times10^{-3}$ | $1.947\times10^{-5}$ | $1.947\times10^{-7}$ |
| | 500 | $1.923\times10^{-2}$ | $1.936\times10^{-4}$ | $1.936\times10^{-4}$ | $1.936\times10^{-8}$ |
| | 5000 | $1.935\times10^{-3}$ | $1.935\times10^{-5}$ | $1.935\times10^{-7}$ | $1.935\times10^{-9}$ |

Table 1 Delay bias versus $P$ and $SNR$ at $D = 0.25$

Although a closed form solution for $\overline{D}$ is not available, numerical values of $\overline{D}$ can be found by using Newton-Raphson method to reveal the effect of different $SNR$ and $P$ on $\overline{D}$. Plot of the delay bias, which is defined as $|\overline{D} - D|$, versus $D$ for $SNR = -10$ dB and $P = 15$ is shown in Figure 2. It can be seen that the bias is largest at $D = 0.2$, which is 0.066. However, the delay error decreases when $SNR$ or $P$ increases and it becomes negligible under high $SNR$ conditions. Moreover, the shape of the plot may change as $SNR$ or $P$ varies. Table 1 illustrates the variation of the delay bias against $SNR$ and $P$ for $D = 0.25$. It is observed that the bias can be reduced by one-tenth approximately by either increasing the $SNR$ by 10 dB or by a ten-fold increase of $P$. This undesirable delay bias in many instances may produce an unacceptable large mean square delay error for the ETDE.

## III. THE ETDGE

It can be easily shown that the optimal weight vector or the Wiener solution is $sinc(i-D)SNR/(1+SNR)$ which is obtained by releasing the constraint imposed on the filter coefficients of the ETDE. Therefore, an improved version of the ETDE is proposed by adding a variable gain control, $\hat{\alpha}(k)$, in series with the delay estimator as depicted in Figure 1, in order to track the factor $SNR/(1+SNR)$ separately so that the minimum MSE can be achieved. In the ETDGE, the error signal, $e(k)$, becomes

$$e(k) = y(k) - \hat{\alpha}(k) \sum_{i=-P}^{P} sinc(i-\hat{D}(k)) x(k-i) \quad (5)$$

Partial differentiating $E\{e^2(k)\}$ with respect to $\hat{\alpha}$ and $\hat{D}$ yields

$$\frac{\partial E\{e^2(k)\}}{\partial \hat{\alpha}} = 2\hat{\alpha}(\sigma_s^2 + \sigma_n^2) \sum_{i=-P}^{P} sinc^2(i-\hat{D})$$

$$-2\sigma_s^2 \sum_{i=-P}^{P} sinc(i-D) sinc(i-\hat{D}) \quad (6a)$$

$$\frac{\partial E\{e^2(k)\}}{\partial \hat{D}} = 2\hat{\alpha}\sigma_s^2 \sum_{i=-P}^{P} sinc(i-D) f(i-\hat{D})$$

$$-2\hat{\alpha}^2(\sigma_s^2 + \sigma_n^2) \sum_{i=-P}^{P} sinc(i-\hat{D}) f(i-\hat{D}) \quad (6b)$$

Putting (6) to zero, it is easy to see that the global minimum occurs when $\hat{\alpha} = SNR/(1+SNR)$ and $\hat{D} = D$. That means the delay estimate of the ETDGE is unbiased for any finite $P$ and is independent of the $SNR$.

While $\hat{D}(k)$ is adjusted according to (3), the gain parameter of the ETDGE is adapted explicitly and independently at the same time. The updating rule for $\hat{\alpha}(k)$ is given by

$$\hat{\alpha}(k+1) = \hat{\alpha}(k) + 2\mu_\alpha e(k) \sum_{i=-P}^{P} x(k-i) sinc(i-\hat{D}(k)) \quad (7)$$

where $\mu_\alpha$ is the step size for $\hat{\alpha}(k)$.

Substituting (1) and (5) into (3) and taking the expected value, we obtain

$$E\{\hat{D}(k+1)\} =$$

$$E\{\hat{D}(k)\} - 2\mu\sigma_s^2 E\left\{\sum_{i=-P}^{P} sinc(i-D)f(i-\hat{D}(k))\right\}$$

$$+2\mu(\sigma_s^2+\sigma_n^2)E\left\{\hat{\alpha}(k)\sum_{i=-P}^{P} sinc(i-\hat{D}(k))f(i-\hat{D}(k))\right\} \quad (8)$$

Since $\sum_{i=-P}^{P} sinc(i-D)f(i-\hat{D}(k)) \rightarrow f(D-\hat{D}(k))$ when $P$ tends to infinity and $f(0) = 0$, therefore

$$E\{\hat{D}(k+1)\} \approx E\{\hat{D}(k)\} - 2\mu\sigma_s^2 E\{f(D-\hat{D}(k))\} \quad (9)$$

Thus the learning characteristics of the delay estimate for the ETDGE is the same as that of the ETDE [4] and is unaffected by $\hat{\alpha}(k)$ when $P$ is large. The convergence behaviour of $\hat{D}(k)$ is given as

$$E\{\hat{D}(k)\} = D + (\hat{D}(0)-D)(1-2\mu\sigma_s^2\pi^2/3)^k \quad (10)$$

where $\hat{D}(0)$ is the initial delay estimate. When $\hat{D}(k)$ converges to $D$, the learning trajectory of $E\{\hat{\alpha}(k)\}$ can be shown to be

$$E\{\hat{\alpha}(k)\} = \alpha + (\hat{\alpha}(0)-\alpha)(1-2\mu_\alpha(\sigma_s^2+\sigma_n^2))^k \quad (11)$$

where $\alpha = SNR/(1+SNR)$ and $\hat{\alpha}(0)$ is the initial gain value which lies between 0 and 1.

To evaluate the performance of the ETDGE, we follow the tedious derivations in [5] to obtain the delay variance, $var(\hat{D})$, which is given by

$$var(\hat{D}) \approx \frac{\mu\sigma_s^2(1+2SNR)}{SNR^2} \quad (12)$$

Comparing $var(\hat{D})$ with the variance of the ETDE, which is approximated by $2\mu\sigma_s^2(1+SNR)/SNR^2$ [5], it can be seen that they are identical and equal to $2\mu\sigma_s^2/SNR$ when $SNR >> 1$. However, when $SNR << 1$, the delay variance of the ETDGE equals to $\mu\sigma_s^2/SNR^2$, which is only half of that of the ETDE.

Now replace $D$ by $D(k) = D + \beta k$ where $\beta$ is the Doppler time compression, and using (9), we can obtain the convergence dynamics for a linearly time-varying delay [6],

$$E\{\hat{D}(k)\} \approx D + \beta k - \frac{3\beta}{2\mu\sigma_s^2\pi^2}$$

$$+\left(\hat{D}(0) - D + \frac{3\beta}{2\mu\sigma_s^2\pi^2}\right)\left(1-\frac{2}{3}\mu\sigma_s^2\pi^2\right)^k \quad (13)$$

Upon convergence, the last term vanishes and $E\{\hat{D}(k)\}$ lags $D(k)$ by the third term which is directly proportional to $\beta$ and inversely proportional to $\mu$ and $\sigma_s^2$. The mean square error of $\hat{D}(k)$, $\varepsilon(\hat{D})$, is equal to the delay variance plus the square of the time lag, which is given by

$$\varepsilon(\hat{D}) \approx \frac{\mu\sigma_s^2(1+2SNR)}{SNR^2} + \left(\frac{3\beta}{2\mu\sigma_s^2\pi^2}\right)^2 \quad (14)$$

Notice that it should be smaller than the mean square error of the ETDE [6] since the ETDGE has a smaller delay variance.

## IV. SIMULATION RESULTS AND CONCLUSIONS

Simulation tests have been carried out to evaluate the performance of the ETDGE for both static and nonstationary delays. In our experiments, the sequences $s(k)$, $n_1(k)$ and $n_2(k)$ are produced by a random number generator of Gaussian distribution with a white spectrum. The signal source has unity power and different $SNR$s are obtained by proper scaling of the random noise sequences. To demonstrate that the ETDGE is unbiased for all filter lengths, $P$ is chosen to be 15 in the ETDE but it is reduced to 3 in the ETDGE. The initial values of the gain and delay parameters are arbitrary set to 1 and 0 respectively. The results provided are the averages of 200 independent runs.

Figure 3 compares the convergence characteristics of the ETDE and the ETDGE when $D(k) = 0.3$, $SNR = -10$ dB and all step sizes were assigned to be 0.00002. As expected, both algorithms gave the same delay adaptation rate and they converged approximately after 30000 iterations. The delay estimate obtained by the ETDGE was very close to the optimal value of 0.3 whereas that of the ETDE was 0.35, which had a bias of about 0.05 and was confirmed by the results as shown in Figure 2. The gain parameter also converged to the desired value, 0.09, which corresponded to a $SNR$ of −10 dB. Furthermore, the variances of the delay estimate of the ETDE and the ETDGE were found to be 0.0033 and 0.0014 respectively. This illustrates that a much smaller delay variance can be achieved by using the ETDGE.

The tracking performance of the ETDGE for $\hat{D}(k)$ and $\hat{\alpha}(k)$ are shown in Figure 4 and Figure 5 respectively. In this test, the actual delay was a linearly time-varying function which was given by $D(k) = 0.25 + 0.0001k$. The $SNR$ was initially set at 10 dB and then step-changed to −10 dB at the 5000th iteration. In order to cope with the nonstationary environment, $\mu$ and $\mu_\alpha$ were selected to a larger value of 0.0002. It can be seen in Figure 4 that after approximately 2000 iterations, the delay estimate converged and lagged $D(k)$ by 0.087 at $SNR$ = 10 dB and 0.11 at $SNR$ = −10 dB. Due to the approximations in deriving (13), discrepancies of 0.011 and 0.034 from the theoretical value are found when $SNR$ = 10 dB and $SNR$ = -10 dB respectively. The delay variances measured at the high $SNR$ and low $SNR$ conditions were found to be 0.0073 and 0.037 respectively. In Figure 5, $\hat{\alpha}(k)$ converged to 0.91 and 0.09, which were their desired values, at the 4000th and the 7000th iterations respectively. From (11), it is observed the time constant of the gain estimate decreases as the noise power increases. This was validated since the learning rate of $\hat{\alpha}(k)$ at $SNR = -10$ dB was much faster than that in the high $SNR$ condition.

In conclusion, the ETDE is shown to be a biased estimator and the corresponding delay error decreases as the number of filter taps or the $SNR$ increases. By adding an adaptive gain control to the ETDE, an unbiased delay estimate for all practical filter lengths can be achieved with a smaller delay variance. Theoretical performance for both static and linearly time-varying delay estimates are given and corroborated by simulations.


## ACKNOWLEDGEMENT

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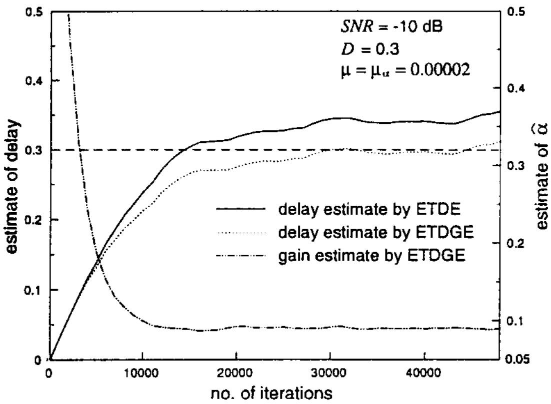


Figure 3 Performance of the ETDE and the ETDGE for a static delay

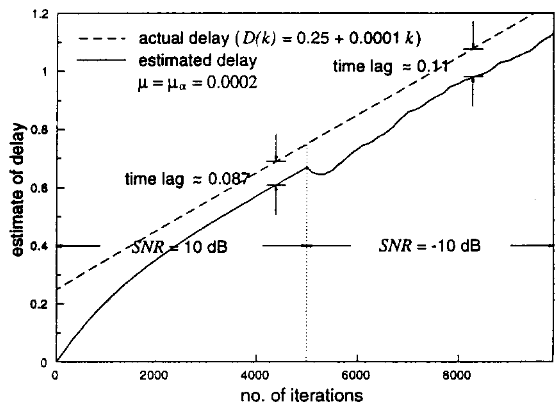


Figure 4 Delay estimate of the ETDGE for a linearly time-varying delay

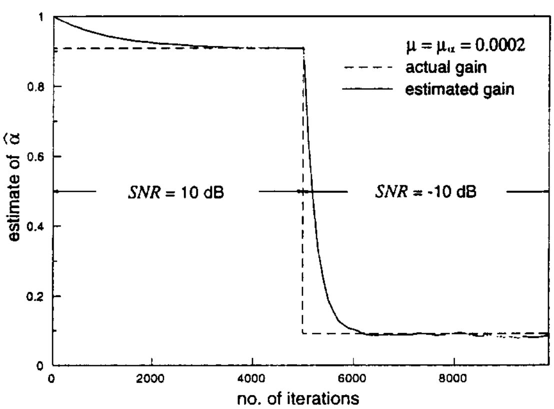


Figure 5 Gain estimate of the ETDGE in a nonstationary signal/noise power environment