Performance Analysis Of Integrated Polyspectrum Based Time Delay Estimators

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ABSTRACT

The problem of estimating the difference in arrival times of a non-Gaussian signal at two spatially separated sensors is considered. The signal is assumed to be corrupted by spatially correlated Gaussian (or a class of non-Gaussian) noise of unknown cross-correlation. We analyze the asymptotic performance of some recently proposed differential time-delay estimators which exploit the integrated polyspectrum of the measurements. The proposed estimators are asymptotically maximum-likelihood when attention is confined to the integrated polyspectra of the measurements. Therefore, the performance of the estimators approaches the Cramer-Rao (CR) bound asymptotically. Expressions for the relevant CR bound are derived. Computer simulations are presented comparing actual performance with the CR bounds for a simple example.

1 Introduction

The estimation of time delay between received signals at two (or more) sensor locations remains an important task in several fields such as sonar, radar, biomedicine, and geophysics [1]-[6]. In passive sonar e.g., the time delay is used to estimate the position and the velocity of a detected acoustic source.

Various methods have been proposed and implemented over the years for time delay estimation [1]-[6]. In [6] we presented two new frequency-domain approaches for differential time-delay estimation using bispectrum or integrated bispectrum. The objective of this paper is to analyze the performance of the estimators of [6]. We also provide some modifications and corrections to [6]. Compared to the time-domain approaches of [2] and [3], [6] does not need the input to be a linear process for consistency to hold true. Consistency of the time-domain approaches of [1] remains unproven, in general [2]. Compared to the bispectrum-based frequency-domain approaches of [1], our approaches of [6] are asymptotically optimal in that we also exploit statistics of the bispectrum (or integrated bispectrum) unlike [1]. Same comments apply when comparisons are made with [5].

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2 Model Assumptions

Let $\{x(k)\}$ and $\{y(k)\}$ denote the (discrete time) measurements at the two sensors. Let $\{s(k)\}$ denote the (non-Gaussian) signal and let $n_i(k)$ $(i = 1, 2)$ be the additive colored noises at the respective sensors. Thus we have

$$x(k) = s(k) + n_1(k), \tag{1}$$

$$y(k) = s(k + D) + n_2(k), \tag{2}$$

where D is the differential time delay (or advance) between the signals at the two sensors. In the above equations, k is an integer and the delay D is a real number. It is assumed that all of the processes involved (i.e., $x(k)$, $y(k)$, $n_1(k)$, and $n_2(k)$) are zero-mean and jointly stationary. The signal $s(k)$ is assumed to be non-Gaussian such that its bispectrum is nonvanishing. The noise processes $\{n_1(k)\}$ and $\{n_2(k)\}$ are independent of the signal $\{s(k)\}$, and are such that their (joint) bispectrum vanishes. For instance, the noise processes may be jointly Gaussian. More precisely, conditions (AS1)-(AS3) of [6] are assumed to hold true for model (1)-(2).

The objective is to estimate the delay D given a data record $\{x(k),\ y(k),\ 1 \leq k \leq N\}$.

Consider the triple correlation function $C_{xxy}(i, k) := E\{x(t+i)x(t+k)y(k)\}$. Denote the cross-bispectrum of input/output (two-dimensional discrete Fourier transform of $C_{xxy}(i, k)$) by $B_{xxy}(\omega_1, \omega_2)$. Similarly, let $B_{xxx}(\omega_1, \omega_2)$ denote the bispectrum of the input process $\{x(k)\}$. We assume that the bispectra of the noise processes are zero and that the noise processes are statistically independent of the $\{u(k)\}$ as well as $\{s(k)\}$. It is also assumed that $B_{sss}(\omega_1, \omega_2) \not\equiv 0$. The cross-spectrum between $\{x^2(k)\}$ and $\{y(k)\}$ is given by $S_{x^2y}(\omega) = \sum_{k=-\infty}^{\infty} C_{xxy}(k, k)\exp\{-j(\omega k)\}$ leading to $S_{x^2y}(\omega) = H^*(e^{j\omega})S_{x^2x}(\omega)$, where $H(e^{j\omega}) = \exp(j\omega D)$ and H^* is the complex conjugate of H. It is easy to show that[4] $S_{x^2x}(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} B_{xxx}(\omega, \omega_2)\, d\omega_2$; hence, the name **integrated bispectrum (polyspectrum)** for $S_{x^2x}(\omega)$. The integrated cross-trispectrum (IT) is $S_{\hat{x}^3y}(\omega)$ where $\hat{x}^3(t) = x^3(t) - 3\,x(t)\,[\mathrm{E}\{x(t)\}]^2$.

3 Time Delay Estimation

It follows from the above development that if $S_{x^2x}(\omega) \neq 0$, then $H^*(e^{j\omega}) = S_{x^2y}(\omega)[S_{x^2x}(\omega)]^{-1}$.

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Let a consistent estimate $\hat{S}_{x^2y}(n)$ of $S_{x^2y}(\omega)$ be available for $\omega = \omega_n = 2\pi n/N_B$, $1 \le n \le (N_B-1)/2$; similarly for $S_{x^2y}(n)$. For instance these estimates may be obtained from a record length of N samples by first dividing the record into K (non-overlapping) blocks, each block of length N_B samples, so that $N = N_B K$; compute the appropriate cross-periodograms for each block and then average over the K blocks; details are in [6]. Let $H(e^{j\omega}|d)$ denote the transfer function $H(e^{j\omega})$ with the differential time delay fixed at d. Let D denote the true value of d. Using the polyspectral estimates instead of the true quantities, define

$$\hat{H}(e^{j(\omega_m)}) \doteq \hat{S}^*_{x^2y}(m)[\hat{S}^*_{x^2x}(m)]^{-1}. \tag{3}$$

Using the results of [8, Sec. 7.4, particularly Problem 7.10.8], it follows for large N (such that both N_B and K become large) that the real and the imaginary parts of the estimate $\hat{S}_{x^2y}(m)$ $(m \neq 0)$ are bivariate Gaussian, and $\hat{S}_{x^2y}(m)$ is independent of $\hat{S}_{x^2y}(n)$ $m \neq n$ $(m, n = 1, 2, \cdots, \frac{N_B}{2} - 1)$, such that

$$\mathrm{E}\{\hat{S}_{x^2y}(m)\} = S_{x^2y}(\omega_m) + O(N_B^{-1}), \tag{4}$$

$$\mathrm{var}\{\mathrm{Re}\{\hat{S}_{x^2y}(m)\}\} = \frac{1}{2K}[S_{x^2x^2}(\omega_m)S_{yy}(\omega_m) + \mathrm{Re}\{S^2_{x^2y}(\omega_m)\}] + O(N^{-1}), \tag{5}$$

$$\mathrm{var}\{\mathrm{Im}\{\hat{S}_{x^2y}(m)\}\} = \frac{1}{2K}[S_{x^2x^2}(\omega_m)S_{yy}(\omega_m) - \mathrm{Re}\{S^2_{x^2y}(\omega_m)\}] + O(N^{-1}), \tag{6}$$

$$\mathrm{cov}\{\mathrm{Re}\{\hat{S}_{x^2y}(m)\}, \mathrm{Im}\{\hat{S}_{x^2y}(m)\}\} = \frac{1}{2K}\mathrm{Im}\{S^2_{x^2y}(\omega_m)\} + O(N^{-1}). \tag{7}$$

Similar results hold for $\hat{S}_{x^2x}(m)$. Using Cor. 7.4.3 of [8] it also follows that for large N (such that $K \to \infty$ and $N_B \to \infty$),

$$\mathrm{cov}\{\hat{S}_{x^2y}(m), \hat{S}_{x^2x}(m)\} = \frac{1}{K}S_{x^2x^2}(\omega_m)S_{x^2y}(\omega_m) + O(N^{-1}), \tag{8}$$

$$\mathrm{cov}\{\hat{S}_{x^2y}(m), \hat{S}^*_{x^2x}(m)\} = \frac{1}{K}S_{x^2x}(\omega_m)S_{x^2y}(\omega_m) + O(N^{-1}), \tag{9}$$

where $\mathrm{cov}\{X, Y\} = E\{XY^*\} - E\{X\}E\{Y^*\}$. For distinct frequencies in $(0, \pi)$, the above covariances are $O(N^{-1})$.

The above results prove useful in establishing Lemma 1 which is a corrected version of [6, Lemma 3]. (The error in [6] lies in ignoring the correlation between the real and the imaginary parts of the cross-spectrum estimates.)

Lemma 1. As $N \to \infty$, the following results are true for any fixed $1 \le m \le \frac{N_B}{2} - 1$ with $\omega_m = 2\pi m/N_B$.

(A) $\sqrt{K}[\hat{H}(e^{j\omega_m}) - H(e^{j\omega_m}|D)]$ converges in distribution to the complex normal distribution $\mathcal{N}_c(0, \sigma_m^2)$ where $\sigma_m^2 = \beta(m)S_{x^2x^2}(\omega_m)S_{yy}(\omega_m)|S_{x^2x}(\omega_m)|^{-2}$, $\beta(m) = 1 + |\alpha_m|^2(S_{x^2x^2}(\omega_m)/S_{yy}(\omega_m)) - 2\mathrm{Re}\{\alpha_m(S_{x^2y}(\omega_m)/S_{yy}(\omega_m))\}$ and $\alpha_m = S^*_{x^2y}(\omega_m)/S^*_{x^2x}(\omega_m)$.

(B) $\hat{H}(e^{j\omega_m})$ and $\hat{H}(e^{j\omega_{m'}})$ are statistically independent for $m \neq m'$. •

Sketch of Proof: A perturbation (Taylor series) expansion of $\hat{H}(e^{j(\omega_m)})$ yields (see also proof of Theorem 8.7.1 of [8])

$$\hat{H}^*(e^{j\omega_m}) = H^*(e^{j\omega_m}|D) + S^{-1}_{x^2x}(\omega_m)\left(\hat{S}_{x^2y}(m) - S_{x^2y}(\omega_m)\right) - H^*(e^{j\omega_m}|D)S^{-1}_{x^2x}(\omega_m)\left(\hat{S}_{x^2x}(m) - S_{x^2x}(\omega_m)\right) + \cdots \tag{10}$$

Using (5)-(9) we can then establish that for large N, $E\{K[\hat{H}(e^{j\omega_m}) - H(e^{j\omega_m}|D)]^2\} = 0$ and $E\{K|\hat{H}(e^{j\omega_m}) - H(e^{j\omega_m}|D)|^2\} = \sigma_m^2$. Part (A) of the lemma then follows by using results from Sec. 4.2 and Theorem P5.2 of [8]. Part (B) similarly follows from the discussion preceding (4) and [8, Thm. P5.2]. □

In [6] it was proposed to estimate time delay d by minimizing the cost

$$J_N(d) = \sum_{m=1}^{(N_B/2)-1} \left| \frac{\hat{S}^*_{x^2y}(m)}{\hat{S}^*_{x^2x}(m)} - e^{j\omega_m d}\right|^2 / \hat{\sigma}_m^2 \tag{11}$$

where $\hat{\sigma}_m^2$ is obtained by replacing all the desired quantities in σ_m^2 by their consistent estimates.

4 Performance Analysis

4.1 Transfer Function Matching

The probability density function (PDF) of the asymptotically complex Gaussian vector $\{\hat{H}(e^{j\omega_m}),\ 1 \le m \le (N_B/2) - 1\}$ assuming d to be the true time delay is given by

$$f(\mathbf{H}|d) = \prod_{m=1}^{(N_B/2)-1} \frac{1}{\pi\sigma_m^2}\exp\left[-\frac{\left|\hat{H}(e^{j\omega_m}) - e^{j\omega_m d}\right|^2}{\sigma_m^2/K}\right]. \tag{12}$$

The cost function used in [6] is $-ln(f(\mathbf{H}|d))$ with σ_m^2 replaced with its consistent estimate $\hat{\sigma}_m^2$. The estimate obtained by maximizing $f(\mathbf{H}|d)$ is therefore the (Gaussian) maximum likelihood estimate, hence, it can be shown to be asymptotically efficient with its asymptotic variance equaling the Cramer-Rao (CR) bound [7]. The variance of the estimator $\hat{d}_N$ of d for large N is therefore given by

$$\mathrm{var}(\hat{d}_N) = E\{(\hat{d}_N - D)^2\}$$

$$= \left| \left[E\left\{ \left[\frac{\partial ln(f(\mathbf{H}|d))}{\partial d} \right]^2 \right\} \right]^{-1} \right|_{d=D} \quad (13)$$

where $E\{\hat{d}_N\} = D$. We have

$$\frac{\partial ln(f(\mathbf{H}|d))}{\partial d} = \sum_{m=1}^{(N_B/2)-1} \frac{2K}{\sigma_m^2} \times$$

$$\mathrm{Re}\left\{ j\omega_m e^{j\omega_m d} [\hat{H}^*(e^{j\omega_m}) - e^{-j\omega_m d}] \right\}. \quad (14)$$

Now exploit Lemma 1 and the properties of complex Gaussian random variables to deduce that

$$E\left\{ \left[\frac{\partial ln(f(\mathbf{H}|d))}{\partial d} \right]^2 \right\} = 2N \left[N_B^{-1} \sum_{m=1}^{(N_B/2)-1} \frac{\omega_m^2}{\sigma_m^2} \right]$$

$$\longrightarrow 2N \left[\frac{1}{2\pi} \int_0^{\pi} \frac{\omega^2}{\sigma^2(\omega)} d\omega \right] \quad \text{as } N_B \to \infty \quad (15)$$

where $\sigma^2(\omega)$ is given by the expression for σ_m^2 (see Lemma 1(A)) with ω_m replaced with ω throughout. In general, one has to calculate the bound (15) numerically.

Minimization of (11) (or maximization of (12)) requires either nonlinear iterative optimization or (as in [6]) computation of a criterion (see Eqn. (22) in [6]) for a continuous range of values of d. In [6] it was proposed to calculate this criterion for a discrete set of values of d via zero-padded FFT calculations (interpolation). In this case the resolution is limited by the amount of zero-padding. An alternative closed-form solution (after mod 2π ambiguity removal discussed in Sec. 4.2) is obtained by phase matching which is discussed next.

4.2 Phase Matching

Define $H(e^{j\omega_m}) = |H(e^{j\omega_m})| e^{j\phi(\omega_m)}$ so that for model (1)-(2), $\phi(\omega_m) = \omega_m D$ (mod 2π). Similarly, set $\hat{H}(e^{j\omega_m}) = |\hat{H}(e^{j\omega_m})| e^{j\hat{\phi}(\omega_m)}$. Therefore, we have

$$\hat{\phi}(\omega_m) = \tan^{-1}\left[\frac{\mathrm{Im}\hat{H}(e^{j\omega_m})}{\mathrm{Re}\hat{H}(e^{j\omega_m})} \right]. \quad (16)$$

The following result is immediate using Lemma 1 and [8, Thm. P5.2].

Lemma 2. As $N \to \infty$, the following results are true for any fixed $1 \le m \le \frac{N_B}{2} - 1$ with $\omega_m = 2\pi m / N_B$.

(A) $\sqrt{K}[\hat{\phi}(\omega_m) - \omega_m D \text{ (mod } 2\pi)]$ converges in distribution to the Gaussian distribution $\mathcal{N}(0, \sigma_{\phi m}^2)$ where $\sigma_{\phi m}^2 = 0.5\sigma_m^2 |H(e^{j\omega_m})|^{-2} = 0.5\sigma_m^2$ and σ_m^2 is as defined in Lemma 1.

(B) $\hat{\phi}(\omega_m)$ and $\hat{\phi}(\omega_{m'})$ are statistically independent for $m \neq m'$. •

It is clear that one must get rid of the mod 2π ambiguity before a phase matching approach can be devised. Suppose that we use the approach of [6] to get an estimate $\bar{D}$ of D to within a resolution of $1/L$ sampling interval (see [6]) where $L = P/N_B$ and $P =$ length of the zero-padded transfer function sequence $\hat{H}(e^{j\omega_m})$. Take $L = 2$ or 4, for instance. Define $\hat{H}'(e^{j\omega_m}) = \hat{H}(e^{j\omega_m}) e^{-j\omega_m \bar{D}}$ leading to

$$\arg\left(\hat{H}'(e^{j\omega_m}) \right) = \hat{\phi}'(\omega_m) = \hat{\phi}(\omega_m) - \omega_m \bar{D}. \quad (17)$$

If $\bar{D}$ is close enough to D, then $\sqrt{K}[\hat{\phi}'(\omega_m) - \omega_m(D - \bar{D})]$ has the same distribution as $\sqrt{K}[\hat{\phi}(\omega_m) - \omega_m D \text{ mod } 2\pi]$.

Now we have a linear model with independent complex Gaussian measurement noise e_m:

$$\hat{\phi}'(\omega_m) = \omega_m(D - \bar{D}) + e_m, \quad 1 \le m \le \frac{N_B}{2} - 1. \quad (18)$$

The so-called Markov (or best linear unbiased) estimate of D is then given by [10, Sec. 4.3]

$$\hat{d}_N = \frac{\sum_{m=1}^{(N_B/2)-1} \omega_m \hat{\phi}'(\omega_m)/\sigma_{\phi m}^2}{\sum_{m=1}^{(N_B/2)-1} \omega_m^2/\sigma_{\phi m}^2} + \bar{D} \quad (19)$$

with the resultant variance [10, Sec. 4.3]

$$\mathrm{var}(\hat{d}_N) = \frac{1}{2N\left[N_B^{-1} \sum_{m=1}^{(N_B/2)-1} \frac{\omega_m^2}{\sigma_m^2} \right]}$$

$$\longrightarrow \frac{1}{2N\left[\frac{1}{2\pi} \int_0^{\pi} \frac{\omega^2}{\sigma^2(\omega)} d\omega \right]} \quad \text{as } N_B \to \infty \quad (20)$$

which is exactly as that in (13) and (15).

5 Simulation Results

The model is given by (1) and (2) with D=5.4. The signal process $\{s(k)\}$ is a zero-mean, i.i.d. one-sided exponentially distributed sequence. Let $\{n(k)\}$ and $\{n'(k)\}$ be two mutually independent, zero-mean, i.i.d. Gaussian sequences each with unit variance. Then we choose

$$n_1(k) = c_1 n(k), \quad (21)$$

$$n_2(k) = c_2[0.9n(k+1) + n'(k)], \quad (22)$$

where the constants c_1 and c_2 are chosen to achieve desired SNR's at the two sensors. The sequences $\{n(k)\}$ and $\{n'(k)\}$ are independent of $\{s(k)\}$.

Fig. 1. Performance Bounds

(Averaged over 100 runs; true delay 5.4)

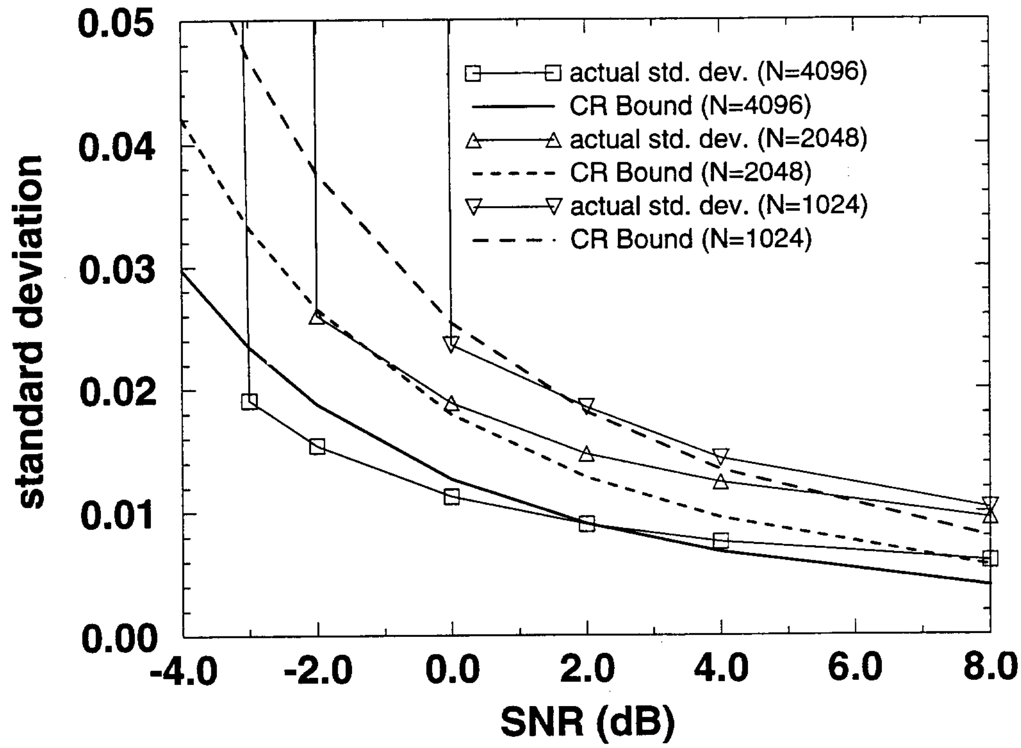

Fig. 1 is based upon 100 Monte Carlo runs. The SNR at the two sensors was kept equal, and was varied from −4dB to +8dB. Experimental standard deviation and the CR bounds are shown in Fig. 1 for three different record lengths. The bounds are based upon (19) (equivalently (15)). The sampled standard deviations are based upon phase-matching. The entire record in each Monte Carlo run was divided into 128 samples long segments with no overlap. It is seen that the performance bounds provide a good indication of the performance above a "threshold" SNR. Below this threshold value of SNR, large estimation errors dominate making the estimate biased and rendering the CR bound useless.

6 References

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