# GLRT DETECTION OF SIGNALS IN REVERBERATION

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## ABSTRACT

A common problem in active sonar is that of multipath propagation in the ocean environment. A generalized likelihood ratio test (GLRT) approach is developed for detecting multiple target returns with unknown time delays and amplitudes in reverberation. The reverberation is modeled by a scattering function that is assumed to be described by the power spectral density of a complex autoregressive process. The generalized likelihood ratio test detector is described and its performance is compared to the clairvoyant optimal processor and an ad hoc processor.

## 1. PROBLEM STATEMENT

A common problem in active sonar is that of multipath propagation in the ocean environment. As a result, target reflections may appear at the receiver at different times, which are, unfortunately, unknown apriori. These returns may or may not overlap in time. To improve detection performance, we could time delay, scale, and phase these returns to coherently combine them. The difficulty in doing so is in the estimation of the amplitudes and time delays of the target returns. To do so, we model the ocean/target as a complex low-pass random linear time invariant filter with impulse response $h[n] = g[n] + \sum_{i=1}^{q} A_i\delta[n-n_i]$.

The impulse response of the filter is assumed to be nonzero over the interval $[0, N_g-1]$. We have decomposed the impulse response into a non-specular component $g[n]$, modeled as a tapped delay line of length $N_g$, and a specular component that is a weighted sum of $q$ Dirac impulses [1]. The nonspecular component characterizes the reverberation, while the specular component represents the target(s). We assume the $g[n]$ are samples from a complex independent Gaussian random process, $g[n] \sim \mathcal{CN}(0, b[n])$, where the $b[n]$, the variance of the tap weights, comprise the sampled range scattering function. The complex amplitudes $A_i$ and the time delays $n_i$ of the target are of interest to us, as are the $b[n]$ so that we may prewhiten the data.

The output of this channel, whose input is the transmitted signal $s[n]$, is

$$x[n] = \sum_{k=0}^{N_g-1} g[k]s[n-k] + \sum_{i=1}^{q} A_i s[n-n_i] + w[n], \quad n = 0, 1, \ldots, N_T-2. \tag{1}$$

The known complex low-pass signal $s[n]$ is assumed to be nonzero over the interval $[0, N_s-1]$ with energy $\mathcal{E} = \sum_{n=0}^{N_s-1} |s[n]|^2 = 1$, and $N_T = N_g + N_s \approx N_g$. Ambient noise, $w[n]$, has been added to the channel output. The $w[n]$ are samples from a complex Gaussian wide sense stationary (WSS) white noise process ($w[n] \sim \mathcal{CN}(0, \sigma_w^2)$), and are independent of the $g[n]$.

Given $N \approx N_T \approx N_g$ samples of $x[n]$, we perform an $N$ point DFT and approximate the linear convolution found in (1) with a circular convolution and write in the frequency domain

$$X[k] = G[k]S[k] + \sum_{i=1}^{q} A_i S[k] e^{-j2\pi k n_i/N} + W[k], \quad k = 0, 1, 2, \ldots, N-1, \tag{2}$$

where the DFT of $x[n]$ is $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}$, $k = 0, 1, 2, \ldots, N-1$. Then, assuming

$$S^2[k] \gg \mathcal{E}\left\{W^2[k]\right\}, \ k = 0, 1, 2, \ldots, N-1 \tag{3}$$

and $S[k] \neq 0$, $k = 0, 1, 2, \ldots, N-1$, we form $Y[k] = X[k]/S[k]$ to yield

$$Y[k] = G[k] + \sum_{i=1}^{q} A_i e^{-j2\pi k n_i/N}, \ k = 0, 1, 2, \ldots, N-1. \tag{4}$$

This is equivalent to a sum of complex sinusoids in colored noise. The frequencies of the complex sinusoids ($f_i = -n_i/N$) in (4) contain the time delay information of interest to us. The $G[k]$ are samples of a zero mean complex Gaussian WSS process. We choose to model $G[k]$ as a complex autoregressive process of known order $p$ (AR(p)) with Power Spectral Density (PSD)

$$b(\lambda) = \frac{\sigma^2}{\left|1 + \sum_{m=1}^{p} a[m]e^{-j2\pi m\lambda}\right|^2}. \tag{5}$$

Here $\lambda$ represents a normalized time delay $\lambda = \Lambda/T_b$, where $\Lambda$ represents time delay and $T_b$ represents the total multipath spread of the channel [2]. We choose $0 \leq \lambda \leq 1$ as the basic period, so that the variance of the tap weights are $b[n] = b(\lambda = \frac{n}{N_g})$, $n = 0, 1, 2, \ldots, N_g - 1$.

The PSD relationship in (5) implies that the $G[k]$ satisfy $G[k] = -\sum_{n=1}^{p} a[n]G[k-n] + u[k]$, where $u[k] \sim \mathcal{CN}(0, \sigma^2)$.

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Hence we have transformed the problem to one of estimating the amplitudes $A_i$ and frequencies $f_i$ of complex sinusoids in complex AR noise with unknown parameters $\mathbf{a} = [a[1]a[2]\dots a[p]]^T$ and $\sigma^2$.

We are now ready to pose the detection problem. It has been transformed to that of detecting a signal in colored noise. Under $\mathcal{H}_0$, when there are no targets present, the data consist of samples from a complex AR(p) process and under $\mathcal{H}_1$ the data consist of the sum of $q$ complex sinusoids in complex AR(p) noise. That is,

$$\begin{aligned} \mathcal{H}_0 &: \; Y[k] = G[k] \\ \mathcal{H}_1 &: \; Y[k] = G[k] + S'[k], \; k = 0, 1, \dots, N-1, \end{aligned} \tag{6}$$

where $S'[k] = \sum_{i=1}^{q} A_i e^{j2\pi k f_i}$.

In this paper the theory of generalized likelihood ratio testing will be used to derive a detector structure for the binary hypothesis problem posed in (6). This approach requires maximum likelihood estimates (MLE) of the unknown parameters under $\mathcal{H}_0$ and $\mathcal{H}_1$. These MLE results may be found in [3].

## 2. GLRT DETECTOR

The GLRT technique is a common method for detecting signals in noise when either or both are characterized by unknown parameters [4]. Given the data $\mathbf{Y} = [[Y[0]Y[1]\dots Y[N-1]]^T$, the GLRT technique forms the modified likelihood ratio

$$l_{GLRT} = \frac{p_1\left(\mathbf{Y}; \hat{\mathbf{A}}, \hat{\mathbf{f}}, \hat{\mathbf{a}}_1, \hat{\sigma}_1^2\right)}{p_0\left(\mathbf{Y}; \hat{\mathbf{a}}_0, \hat{\sigma}_0^2\right)}. \tag{7}$$

where $p_i$ is the PDF under hypothesis $\mathcal{H}_i$, $\hat{\mathbf{A}}$, $\hat{\mathbf{f}}$, $\hat{\mathbf{a}}_1$, and $\hat{\sigma}_1^2$ are the MLE of their respective parameters under $\mathcal{H}_1$, and $\hat{\mathbf{a}}_0$ and $\hat{\sigma}_0^2$ are the MLE of their respective parameters under $\mathcal{H}_0$. The GLRT detector compares $l'_{GLRT} = \ln l_{GLRT}$ to a threshold $\gamma$ and if

$$l'_{GLRT} = \ln p_1 - \ln p_0 > \gamma \;\Rightarrow\; \mathcal{H}_1$$

$$l'_{GLRT} = \ln p_1 - \ln p_0 < \gamma \;\Rightarrow\; \mathcal{H}_0.$$

Substitution of the MLE's of the parameters from [3] into (7) yields $l'_{GLRT} =$

$$N' \max_{\mathbf{f}} \left[ \ln \frac{\mathbf{Y}_p^H \mathbf{P}_H^\perp \mathbf{Y}_p}{\mathbf{Y}_p^H \left[\mathbf{P}_H^\perp - \mathbf{P}_H^\perp \mathbf{E}\left(\mathbf{E}^H \mathbf{P}_H^\perp \mathbf{E}\right)^{-1} \mathbf{E}^H \mathbf{P}_H^\perp\right] \mathbf{Y}_p} \right], \tag{8}$$

where $N' = N - p$, $\mathbf{Y}_p = [Y[p]Y[p+1]\dots Y[N-1]]^T$, $\mathbf{P}_H^\perp = \mathbf{I} - \mathbf{H}\left(\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H$, $\mathbf{E} = [\mathbf{e}_1 \mathbf{e}_2 \dots \mathbf{e}_q]$, $\mathbf{e}_i = \left[e^{j2\pi f_i p} e^{j2\pi f_i (p+1)} \dots e^{j2\pi f_i (N-1)}\right]^T$, and

$$\mathbf{H} = \begin{bmatrix} Y[p-1] & Y[p-2] & \dots & Y[0] \\ Y[p] & Y[p-1] & \dots & Y[1] \\ \vdots & \vdots & \ddots & \vdots \\ Y[N-2] & Y[N-3] & \dots & Y[N-1-p] \end{bmatrix}.$$

Note that (8) requires a search over a $q$-dimensional space.

In general, the performance of a GLRT detector is difficult to obtain, although the asymptotic (large data record) performance of a GLRT detector is related to the asymptotic statistics of the MLE used in the modified likelihood ratio ([4]). Under certain regularity conditions, the asymptotic distribution of the GLRT detector test statistic is chi-square under both hypotheses with different noncentrality parameters. For the current problem, the regularity conditions require that $\mathbf{A}$, $\mathbf{f}$, $\mathbf{a}$, and $\sigma^2$ be estimable under $\mathcal{H}_0$. Unfortunately $\mathbf{f}$ is not. An investigation of this distribution is the subject of an on going study and will be reported on in the future. Also, modifications to the GLRT that yield tractable performance results have been investigated [5]

With an additional assumption, a second GLRT detector can be constructed that does meet the regularity conditions. If in the previous problem, the frequencies in (4) were known (i.e. known target time delays), the development of the test statistic would proceed exactly as above without the need to find the maximum over $\mathbf{f}$ in (8). We will refer to this statistic as $l''_{GLRT}$. That is, for the case of known frequencies (time delays), $l''_{GLRT} =$

$$N' \left[ \ln \frac{\mathbf{Y}_p^H \mathbf{P}_H^\perp \mathbf{Y}_p}{\mathbf{Y}_p^H \left[\mathbf{P}_H^\perp - \mathbf{P}_H^\perp \mathbf{E}\left(\mathbf{E}^H \mathbf{P}_H^\perp \mathbf{E}\right)^{-1} \mathbf{E}^H \mathbf{P}_H^\perp\right] \mathbf{Y}_p} \right]. \tag{9}$$

It can be shown that the asymptotic performance of $2\,l''_{GLRT}$ is chi-square with $2q$ degrees of freedom (DOF) [4]. Under $\mathcal{H}_0$, the distribution is central, and under $\mathcal{H}_1$ it is noncentral with noncentrality parameter $\beta_{GLRT}$. That is, $\mathcal{H}_0 : 2\,l''_{GLRT} \sim \chi^2(2q, 0)$, and $\mathcal{H}_1 : 2\,l''_{GLRT} \sim \chi^2(2q, \beta_{GLRT})$. The noncentrality parameter is a measure of the discrimination between the two hypotheses and is related to the Fisher information matrix (FIM) of the real parameter vectors $\Theta_r = [Re[\mathbf{A}^T]\; Im[\mathbf{A}^T]]^T$, which characterizes the signal, and $\Theta_s = [Re[\mathbf{a}^T]\; Im[\mathbf{a}^T]\; \sigma^2]^T$, which characterizes the noise [4]. Since we are addressing a signal in noise problem where the signal characterizes the mean of the data and the noise characterizes the covariance structure, the FIM decouples into block diagonal form [6] to yield

$$\beta_{GLRT} = \Theta_r^T \mathbf{I}_{\theta_r\theta_r}(\mathbf{0}, \Theta_s)\Theta_r, \tag{10}$$

where

$$\mathbf{I}_{\theta_r\theta_r}(\Theta_r, \Theta_s) = \mathcal{E}\left\{ \left(\frac{\partial \ln p_1}{\partial \Theta_r}\right) \left(\frac{\partial \ln p_1}{\partial \Theta_r}\right)^T \right\} \tag{11}$$

and where $\mathbf{I}_{\theta_r\theta_r}(\Theta_r, \Theta_s)$ is evaluated $\Theta_r = \mathbf{0}$ and the true values of $\Theta_s$ in (10). Note that the decoupling of the FIM implies that the asymptotic performance of the GLRT detector in (9) is equivalent to the clairvoyant detector, i.e., the detector that uses perfect knowledge of the noise parameter vector $\Theta_s$ [7].

## 3. GLRT PERFORMANCE - KNOWN TIME DELAY

In this section we compare the performance of the GLRT detector in (9) with two optimal detectors for the case of a single target ($q = 1$) located at a known time delay $n_1$ (known frequency $f_1 = -n_1/N$). The sampled range scattering function, $b[n]$ , $n = 0, 1, 2, \ldots, N_g - 1$ for $N_g = 400$ shown in Fig. 1 uses an AR(7) model derived from analysis of actual in-water data. The correlation function of the linear FM transmit signal used in this study is shown in Fig. 2.

The clairvoyant optimal matched filter (COMF) assumes that the receive signal is known exactly and the covariance of the noise is known. The clairvoyant optimal incoherent matched filter (COIMF) is equivalent to the COMF except that the complex amplitude of the target is assumed to be unknown.

If knowledge of $A_1, n_1$, and the noise covariance is available (i.,e., clairvoyant), the binary hypothesis problem reduces to that of detecting a known signal in colored Gaussian noise with known covariance matrix. Using (1) and (3), we can pose the detection problem in the time domain as,

$$\begin{aligned} \mathcal{H}_0 &: \mathbf{x} = \mathbf{C}_s\mathbf{g} \\ \mathcal{H}_1 &: \mathbf{x} = \mathbf{C}_s\mathbf{g} + A_1\mathbf{T}_s, \end{aligned} \tag{12}$$

where $\mathbf{x} = [x[0]x[1]\ldots x[N-1]]^T$, $\mathbf{g} = [g[0]g[1]\ldots g[N-1]]^T$, $\mathbf{C}_s$ is the $(N \times N)$ nonsingular convolution matrix of the transmit signal, and $\mathbf{T}_s$ is formed from the vector $\mathbf{s} = [s[0]s[1]\ldots s[N_s-1]]^T$ by $\mathbf{T}_s = [\mathbf{0}_{n_1}^T\ \mathbf{s}^T\ \mathbf{0}_{N-N_s-n_1}^T]^T$ where $\mathbf{0}_n$ is an n-dimensional vector of zeros.

Under $\mathcal{H}_0$ and $\mathcal{H}_1$, $\mathbf{x}$ is a complex Gaussian random vector; $\mathcal{H}_0 : \mathbf{x} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}_x)$, $\mathcal{H}_1 : \mathbf{x} \sim \mathcal{CN}(A_1\mathbf{T}_s, \mathbf{R}_x)$. Here $\mathbf{R}_x$, the covariance of $\mathbf{x}$, is $\mathbf{R}_x = \mathcal{E}\left\{\mathbf{C}_s\mathbf{g}\mathbf{g}^H\mathbf{C}_s^H\right\} = \mathbf{C}_s\mathbf{R}_g\mathbf{C}_s^H$, where $\mathbf{R}_g = \mathcal{E}\left\{\mathbf{g}\mathbf{g}^H\right\}$ is a diagonal matrix whose main diagonal consists of $b[0], b[1], \ldots, b[N-1]$. We will assume $b[n] > 0\ \forall\ n$. Therefore, $\mathbf{R}_x$ is positive definite.

The well known solution is to prewhiten the noise and then implement a filter matched to the signal at the output of the prewhitener [1]. The test statistic is

$$l_{COMF} = 2Re\left[A_1^*\mathbf{T}_s^H\mathbf{R}_x^{-1}\mathbf{x}\right]. \tag{13}$$

Given $l_{COMF}$ is found via a linear operation on $\mathbf{x}$, it is a real Gaussian scalar random variable, and it can be shown [6] that $\mathcal{H}_0 : l_{COMF} \sim \mathcal{N}\left(0, \sigma_{COMF}^2\right)$,
$\mathcal{H}_1 : l_{COMF} \sim \mathcal{N}\left(\mu_{COMF}, \sigma_{COMF}^2\right)$, where $\mu_{COMF} = \sigma_{COMF}^2 = 2|A_1|^2\mathbf{T}_s^H\mathbf{R}_x^{-1}\mathbf{T}_s$.

It may be unrealistic to assume that one would ever know the complex amplitude of the specular return. In this case it can be shown [1] that the optimum processor is the COIMF that forms the test statistic

$$l_{COIMF} = \left|\mathbf{T}_s^H\mathbf{R}_x^{-1}\mathbf{x}\right|^2. \tag{14}$$

If $l_{COIMF}$ is normalized by the known quantity $\mathbf{T}_s^H\mathbf{R}_x^{-1}\mathbf{T}_s$, the resulting statistic is $\chi^2$ with two DOF (a derivation with real varibles can be found in [8]). Under $\mathcal{H}_0$ it is central and under $\mathcal{H}_1$ it is noncentral with noncentrality parameter

$$\beta_{COIMF} = 2|A_1|^2\mathbf{T}_s^H\mathbf{R}_x^{-1}\mathbf{T}_s. \tag{15}$$

As noted in section 2, the GLRT detector in (9) is asymptotically equivalent to the detector that uses perfect knowledge of the noise parameters, therefore it is equivalent to the COIMF and $\beta_{GLRT} = \beta_{COIMF}$.

For a fixed value of $P_{fa} = 0.001$, the probability of detection for each of the techniques was computed as the time delay of the specular return was allowed to range from $n_1 = 1$ to $n_1 = N_g$. With $A_1 = 1$, the results are shown in Fig. 3.

## 4. GLRT PERFORMANCE - UNKNOWN TIME DELAY

In the previous section we investigated the asymptotic performance of the GLRT detector when the location of the specular return was known. As noted in Section 2, the general performance of the GLRT detector (8) ,when the location of the specular return(s) is unknown, is very difficult to analyze. However, Monte Carlo techniques can be used to compute performance for specific cases. In this section detection performance of the GLRT technique for the case of a single specular return will be presented. It will be compared to the performance of an ad hoc approach we refer to as the normalized incoherent matched filter (NIMF).

The NIMF approach uses an incoherent matched filter (IMF), and, in an attempt to provide for a constant false alarm rate (CFAR), performs a normalization on the IMF output. The IMF forms the test statistic [8]

$$l_{IMF}(n_1) = \left|\mathbf{s}^H\mathbf{x}_{n_1}\right|^2, \tag{16}$$

where $\mathbf{x}_{n_1} = [x[n_1]x[n_1+1]\ldots x[n_1+N_s-1]]^T$. The NIMF then forms the test statistic

$$l_{NIMF}(n_1) = \frac{l_{IMF}(n_1)}{l'_{IMF}}, \tag{17}$$

where, $l'_{IMF} = \frac{1}{M}\sum_{i=1}^{M} l_{IMF}(m_i)$, and the $l_{IMF}(m_i)$ are computed from data sets $\mathbf{x}_{m_i}$ in the vicinity of $\mathbf{x}_{n_1}$.

Two tests will be described using the sampled scattering function and transmit signal from the previous section. In the first scenario, a specular target return with amplitude $A_1 = \sqrt{15}$ is located at time delay $n_1 = 180$. In the second scenario, a specular target return with amplitude $A_1 = \sqrt{5}$ is located at time delay $n_1 = 300$. The target in the second scenario is located at a time delay at which the scattering function is relatively constant or 'white', while in the first scenario, the scattering function is not constant but 'colored' in the vicinity of the target. In both cases, the NIMF generated the normalization factor using $M = 8$ sets of data. For the first case, it was computed from data sets $\mathbf{x}_{m_i}$ , $m_i = 140, 150, 160, 170, 190, 200, 210, 220$ , and for the second case, the data sets $\mathbf{x}_{m_i}$ , $m_i = 260, 270, 280, 290, 310, 320, 330, 340$ were used. A set of 500 Monte Carlo runs were made for each scenario.

At this point it is important to clearly define what is meant by a correct detection. For the case of unknown time delay(s), both the GLRT and the NIMF compute a set of statistics - one for each possible value of the time delay(s) of the specular return(s) - and choose the maximum value as the test statistic. A detection is declared when the test statistic crosses a threshold. If this occurs under $\mathcal{H}_1$, we may define this to be a correct detection, or we may impose an additional requirement that the threshold crossing occurs at the correct time delay(s). In this investigation, we will use the latter definition and if, under $\mathcal{H}_1$, a threshold crossing occurs at an incorrect time delay, it will be counted as a false alarm.

A receiver operating characteristic (ROC) curve for the GLRT derived from the simulation data for the two cases is shown in Fig. 4. Note that due to the definition of correct detection used in this investigation, a probability of detection of 1 is not achieved in either case. ROC curves for the NIMF for the two cases are shown in the figure. The most important thing to note is that, due to normalization, the NIMF performs poorly relative to the GLRT - particularly from the point of view of the maximum attainable probability of detection.

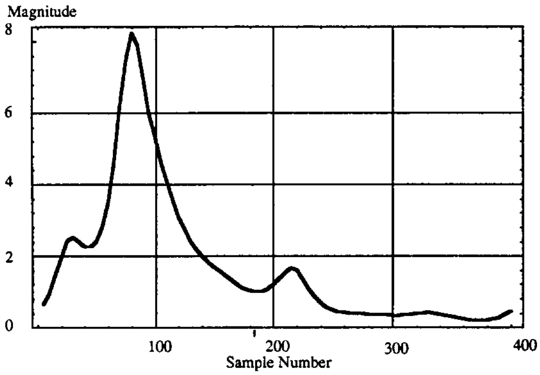


Figure 1: Sampled range scattering function (b[n])

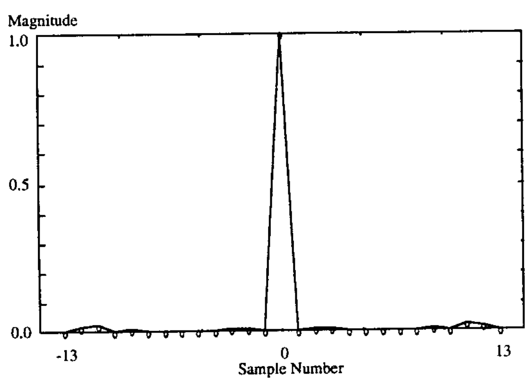


Figure 2: Transmit signal correlation function

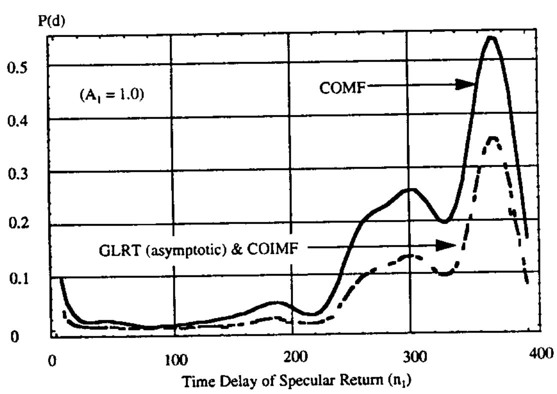


Figure 3: GLRT and COMF performance results

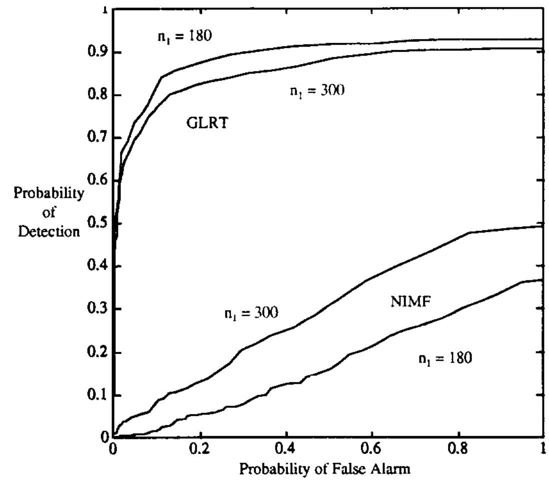


Figure 4: GLRT and NIMF ROC Curves

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