THE PARTITIONED EIGENVECTOR METHOD FOR TOWED ARRAY SHAPE ESTIMATION

J. J. Smith, Y.H. Leung and A. Cantoni

Adaptive Signal Processing Laboratory, Australian Telecommunications Research Institute,
Curtin University of Technology, Bentley W.A. 6102

ABSTRACT

The eigenvector method for estimating the positions of the receivers of an ocean-towed array is based on the eigendecomposition of the array signal correlation matrix to find the phase delays between the array receivers. Previous work has shown that for reasonable SNR, the bias and variance of the phase estimates is relatively independent of the number of receivers in the array. This suggests that the computational cost of the eigenvector method could be substantially improved by partitioning the array receivers into groups of smaller sub-arrays and then applying the eigenvector method to each sub-array. This paper introduces the interleaved partitioned eigenvector method, and shows that it significantly reduces the computational cost without adversely affecting the quality of the position estimates. Numerical examples substantiate the theoretical work of this paper and also demonstrate the improvement in beamforming when employing the shape estimation algorithm.

1. INTRODUCTION

The eigenvector method [1] is an algorithm for estimating the positions of the receivers mounted on a towed flexible array. It is based on the measurement of the phase delays between the array receivers of a narrowband planewave source in the presence of uncorrelated noise. With the knowledge of the wavelength and direction of the source, and using certain physical constraints, the relative phase delays are used to estimate the receiver positions.

The relative phase delays are calculated by performing an eigendecomposition of the correlation matrix of the receiver outputs. The number of computations required is approximately proportional to the cube of the number of array receivers [2]. For arrays with many receivers this computational load could pose a serious problem.

A potentially simple solution to the problem of computational cost is to partition the array into several smaller sub-arrays and to perform the eigenvector method upon each of them. This has the added attraction that with appropriate hardware, the eigendecomposition of each of the smaller correlation matrices can be performed concurrently. However it is necessary to consider the effect of reducing the size of the correlation matrix on the accuracy of the position estimates.

If the correlation matrix is known exactly then the relative phase delays can be calculated exactly and independently of the number of receivers and SNR. However when finite sampling is used to estimate the correlation matrix the statistical performance of the eigenvector method is required. Previous work [3] has shown that position estimation accuracy is indeed relatively independent of the correlation matrix size. In fact, for more than around 6 receivers and reasonable levels of uncorrelated noise and number of sampling snapshots, the improvement of position estimation accuracy with a higher number of receivers, is insignificant. This suggests that array partitioning is a practical solution to the problem of reducing the computational complexity of the eigenvector method.

In [4] two partitioning techniques were introduced for nominally linear arrays : interleaved partitioning and segmented partitioning. The segmented approach divides the array into sub-arrays of consecutive receivers while the interleaved approach physically separates the receivers in each sub-array. The basis behind the interleaved approach is that it will be more robust against spatially correlated noise. This issue is investigated in [4]. This paper concentrates on the interleaved partitioning technique although the statistical analysis is similar for both.

Section 2 describes the interleaved partitioning eigenvector method and gives the reduction in computational requirements for the eigendecomposition.

Section 3 gives an approximation to the bias and variance of the position estimates and then discusses the efficiency of the partitioned and non-partitioned eigenvector methods in the sense of minimizing the variance and bias of the position estimates and the computational complexity.

Section 4 validates the expressions presented in section 3 with numerical simulations. It also gives an example of the improvement in beamforming when using the shape estimation algorithm.

2. THE INTERLEAVED PARTITIONED EIGENVECTOR METHOD

This section briefly describes the theoretical background to the eigenvector method and then shows how it is modified to form the interleaved partitioned eigenvector method.

Finally this section calculates the computational advantage of interleaved partitioning.

2.1 The Eigenvector Method

The correlation matrix of the output of an array with L receivers when there is a single narrowband planewave source impinging upon the array in the presence of uncorrelated noise, is :

$$R = E[x(t)x^H(t)] = \sigma_s^2 s(\theta) s^H(\theta) + \sigma_n^2 I \quad (1)$$

where σ_s^2 is the source power, σ_n^2 is the white noise power observed on one receiver of the array, and $s(\theta)$ is the steering vector in the source direction, θ.

Let $p = [p_1, \ldots, p_L]^T$ denote the eigenvector corresponding to the largest eigenvalue of R. It is shown in [5] that p is proportional to the steering vector of the array in the direction of the source and can therefore be used to calculate the phase change of the source from receiver i to receiver j :

$$\Delta\phi_{i,j} = \arg(p_j) - \arg(p_i) = \phi_j - \phi_i \quad (2)$$

If the shape of the array is relatively linear, the source direction is away from the general direction of the array, and the source wavelength λ is greater than or equal to twice the array receiver spacing r, then the position of the i^{th} receiver is given by [1] :

$$x_i = \cos\theta \sum_{j=2}^{i} d_j + \sin\theta \sum_{j=2}^{i} \sqrt{r^2 - d_j^2} \quad (3)$$

$$y_i = \sin\theta \sum_{j=2}^{i} d_j - \cos\theta \sum_{j=2}^{i} \sqrt{r^2 - d_j^2} \quad (4)$$

where d_j is the phase distance defined by :

$$d_j = \frac{\lambda}{2\pi}\Delta\phi_{(j-1),j} \quad (5)$$

2.2 Interleaved Partitioning

The mechanics of interleaved partitioning can most easily be described by first making two definitions.

- Define the term 'receiver spacing' to be the distance between consecutive receivers in a linear array. Assume that the receiver spacings are all equal.
- Define the term 'interleaved factor' to be the number of receiver spacings between two consecutive receivers in an interleaved sub-array. Assume that the interleaved factors are all equal.

Interleaved partitioning is achieved by dividing an array into sub-arrays in which consecutive receivers have a fixed interleaved factor, S_I. The interleaved sub-arrays have no common receivers. An additional sub-array named the link sub-array, has one receiver in each interleaved sub-arrray. The link sub-array has a receiver spacing of one greater than the interleaved factor. Figure 1 gives an example of interleaved partitioning.

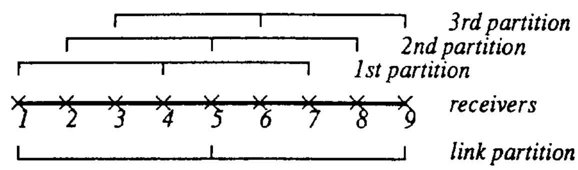

Figure 1. An example of interleaved partitioning with $S_I = 3$ and 9 receivers.

A correlation matrix is calculated for each sub-array and using the eigenvector method, the relative phases between receivers in each sub-array can be calculated. Since the sub-arrays have common receivers (via the link sub-array) it is simple to calculate the relative phases between each pair of consecutive receivers, and apply equations (3) to (5) to get the position estimates.

2.3 Computational Advantage of Partitioning

For an array of L receivers, the computational cost of the eigenvector method is approximately [2] :

$$C_{EM} \propto L^3 \quad (6)$$

For an array partitioned into sub-arrays with interleaved factor S_I, then the computational cost of the interleaved partitioning method is approximately :

$$C_{IP} \propto \sum_{i=1}^{S_I} (M_{I,i})^3 + S_I^3 \quad (7)$$

where $M_{I,i}$ is the number of receivers in the i^{th} sub-array.

Equation (7) can be minimized by selecting S_I as $\lfloor\sqrt{L}\rfloor$. Assuming L is a large square number then $M_{I,i} = S_I$ and the cost becomes approximately :

$$C_{IP} \propto L^2 \quad (8)$$

This represents a computational cost reduction of the order of L over the non-partitioned eigenvector method.

3. POSITION ESTIMATE STATISTICS

In practice, the receiver position estimates are calculated using an estimate of R. This section gives the approximate statistics of the position estimates when the maximum likelihood estimator of R is used, i.e. :

$$\hat{R} = \frac{1}{K}\sum_{k=1}^{K} x(k)x^H(k) \quad (9)$$

where $x(k)$ is a vector representing the k^{th} snapshot of the array output and K is the total number of independent snapshots.

3.1 Bias and Variance of the Position Estimates

In [3] the bias and variance of the i^{th} receiver position estimate are derived in the form of a 2nd order series expansion of the random variables $\hat{d}_2, \ldots, \hat{d}_L$ when equation (9) is used to estimate $\boldsymbol{R}$. They are shown to be accurate if the mean and higher order statistics of $\hat{d}_j$ are small, i.e. the array shape lies in the general vicinity of the x axis and the calibrating source is close to broadside. Since partitioning only affects the statistics of the phases estimates, these expressions apply to both the partitioned and non-partitioned eigenvector methods. The expressions are :

$$\mathrm{Bias}[i] = \left| \sum_{j=2}^{i} \left(r - \frac{E[\hat{d}_j^2]}{2} - \frac{E[\hat{d}_j^4]}{8r^3} - \sqrt{r^2 - d_j^2} \right) \right| \quad (10)$$

$$\mathrm{Var}[i] = \sum_{j=2}^{i} \sum_{k=2}^{i} \left(\mathrm{Cov}[\hat{d}_j, \hat{d}_k] - \frac{E[\hat{d}_j^2 \hat{d}_k^2] - E[\hat{d}_j^2] E[\hat{d}_k^2]}{4r^2} \right) \quad (11)$$

3.2 Phase Distance Statistics

The first and second order statistics of $\Delta\hat{\phi}_{i,j}$ for a single sub-array are derived in [6] using a 2nd order Taylor series expansion. Using these results and (5) gives:

$$E[\hat{d}_j] \approx d_j \qquad \mathrm{Var}[\hat{d}_j] \approx \frac{2\pi}{\lambda} L \beta(L) \quad (12)$$

$$\mathrm{Cov}[\hat{d}_j, \hat{d}_k] \approx \begin{cases} 0 & |j-k| > 1 \\ \dfrac{-\lambda^2 L \beta(L)}{8\pi^2} & |j-k| = 1 \end{cases} \quad (13)$$

$$\text{where } \beta(L) = \frac{\left(L + \frac{1}{\mathrm{SNR}} \right)}{\mathrm{SNR} K L^2} \text{ and } \mathrm{SNR} = \frac{\sigma_s^2}{\sigma_n^2} \quad (14)$$

It is also shown in [6] that with practical values of K, SNR, L, and small values of d_j, the distribution of $\hat{d}_j$ is approximately Gaussian.

To determine the statistics of the phase distance estimates when using the interleaved partitioned eigenvector method, it is necessary to know the covariance between relative phase estimates of different sub-arrays and then use equation (5). The relative phase covariance is derived in [4] resulting as follows.

Let receivers a and b $(a \neq b)$ belong to sub-array x and receivers c and d $(c \neq d)$ belong to sub-array y. Then to a 2nd order approximation :

$$\mathrm{Cov}[\Delta\hat{\phi}_{ab}, \Delta\hat{\phi}_{cd}] \approx \begin{cases} 0 & (a \neq b \neq c \neq d) \\ \dfrac{1}{2K\mathrm{SNR}} & (a = c, b \neq d) \text{ or } (b = d, a \neq c) \\ \dfrac{-1}{2K\mathrm{SNR}} & (a = d, b \neq c) \text{ or } (b = c, a \neq d) \end{cases} \quad (15)$$

3.3 Efficiency of the Partitioned and Non-Partitioned Eigenvector Methods.

Define the efficiency of an array shape estimation method to be [4] :

$$\mathrm{Eff} = \frac{1}{C} \left(L \sum_{i=1}^{L} \mathrm{Var}[i] + (\mathrm{Bias}[i])^2 \right)^{-1/2} \quad (16)$$

where C is the computational cost.

Substituting equations (6), (7), (10) - (15) into (16) would result in complex and not very useful equations. Instead, a numerical example is presented below :

- A 25 receiver array bent into a complete sinusoid with max. displacement of 5m, λ = 6m, r = 3m, source direction = 90 deg (broadside), snapshots = 50, SNR = 0dB.

Partitioning Method	Sub-Array Size	Improvement in Efficiency
None	25	1.0
Interleaved	5	17.7

Table 1. Interleaved partitioning efficiency.

4. NUMERICAL EXAMPLES

Simulations were run to verify the expressions for the receiver position estimation bias and variance using the array scenario listed in section 3.3. Figure 3 shows the theoretical bias and variance of the array receivers when the shape is estimated using both the interleaved partitioned (with S_I=5) and non-partitioned methods. Figure 4 shows the bias and variance when the positions are estimated over 500 simulation trials.

The figures demonstrate that for both the interleaved partitioned and non-partitioned methods, the position bias accumulates down the array, and that the position variance remains relatively constant.

The simulation results also support the theoretical predictions that partitioning does not significantly affect the quality of the position estimates and shows that the second order Taylor series expansion provides sufficient approximation for the position statistics.

A further simulation was run to show the beamformer output power over a range of look directions when using a LCMV beamformer with a single linear constraint in the look direction. A single source at 60 deg. at 0dB SNR was used to calibrate the array shape, and then moved to 105 deg. for the beamforming. Matrix $\boldsymbol{R}$ was estimated using the true shape, and the steering vectors calculated using (i) the true shape, (ii) estimated shape (non-partitioned), (iii) estimated shape (interleaved partitioned), and (iv) a linear shape.

The results, plotted in Figure 2, show that the array shape estimation techniques give a beamformer power output that is similar to that when the true shape is known.

Note that the two shape estimation techniques gave indistinguishable results.

5. CONCLUSION

The expressions for the position estimation statistics show that the quality of the estimates are relatively independent of the number of receivers used to generate the correlation matrix. The paper shows that the interleaved partitioning technique yields similar quality estimates to the non-partitioned eigenvector method at a substantially reduced computational cost.

The theoretical statistics are supported with results from numerical simulations. A further numerical example demonstrates that the interleaved partitioned eigenvector method dramatically improves beamformer output power in the source direction.

6. REFERENCES

[1] D. A. Gray, W. O. Wolfe, and J. L. Riley, "An Eigenvector Method for Estimating the Positions of the Elements of an Array of Receivers", *ASSPA 89*, Apr. 1989.

[2] J.H. Wilkinson, "The Algebraic Eigenvalue Problem", *New York : Oxford University Press*, 1965.

[3] J.J. Smith, Y.H. Leung and A. Cantoni, "Array Shape Estimation : Performance of the Eigenvector Method", *Technical Report ASPL-1993-4, Uni. of West. Australia*, Sept. 1993.

[4] J.J. Smith, Y.H. Leung and A. Cantoni, "The Partitioned Eigenvector Method for Array Shape Estimation", *Technical Report ASPL-1994-1, Uni. of West. Australia*, Feb. 1994

[5] R. O. Schmidt, "Multiple Emitter Location and Signal Parameter Estimation", *IEEE Transactions on Antennas and Propagation*, vol. AP-34 no. 3, March 1986.

[6] J.J. Smith, Y.H. Leung & A. Cantoni, "Statistics of the Phase Delays between Array Receivers Estimated from the Eigendecomposition of the Signal Correlation Matrix", *ICASSP 94*, April 1994.

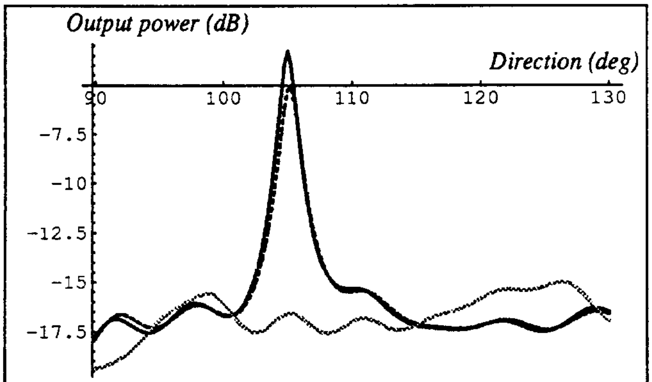

Figure 2. Beamforming with various shape estimates. Solid line :- true shape, grey line :- linear shape, dotted line :- partitioned & non-partitioned eigenvector method shape estimates (indistinguishable).

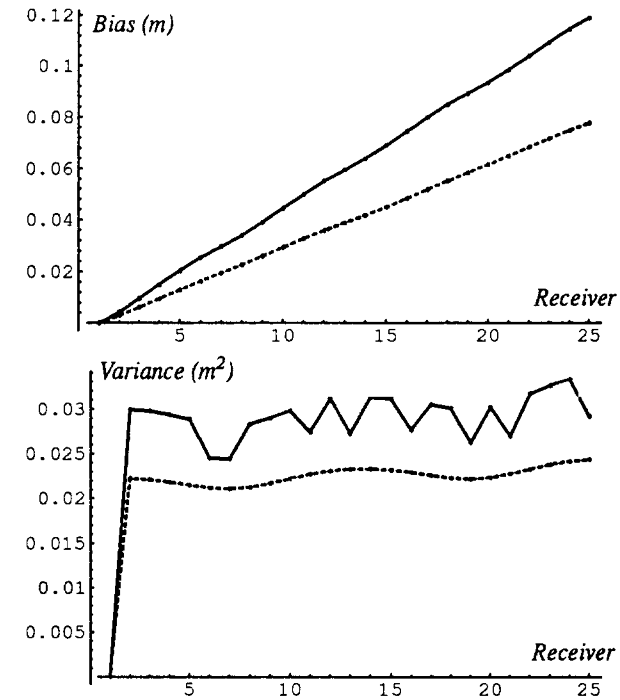

Figure 3. Theoretical bias and variance of the receiver positions. Solid line :- interleaved partitioned method, dashed line :- non-partitioned method

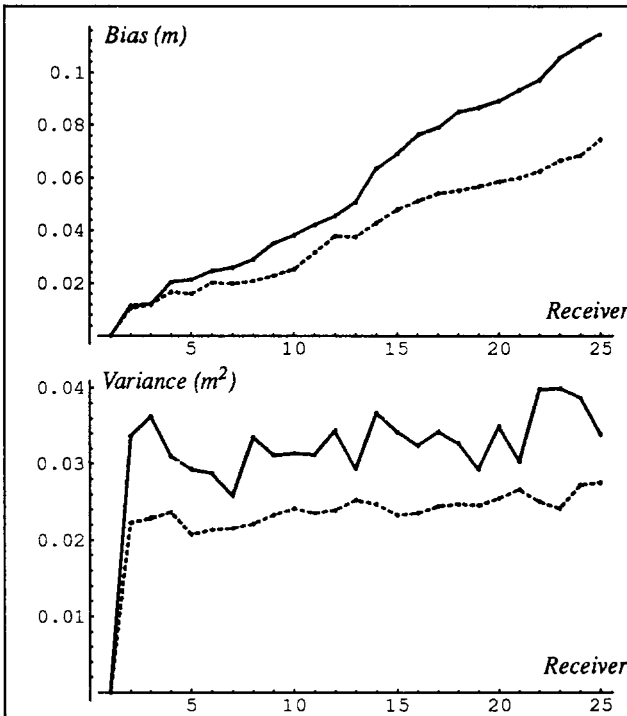

Figure 4. Simulated bias and variance of the receiver positions. Solid line :- interleaved partitioned method, dashed line :- non-partitioned method