GROUND PENETRATING RADAR TOMOGRAPHY

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ABSTRACT

We present a survey of theory and practice in tomographic imaging of subsurface features utilizing ground penetrating radar (GPR). The discussion will include: *i.)* A brief review of the equations governing radar scattering culminating in the Lippmann Schwinger (LS) equation. *ii.)* A summary of approximations often made in developing GPR imaging algorithms based on inversion of the LS equation: the scalar wave approximation, the two dimensional scattering model, and neglect of multiple scattering. *iii.)* A description of certain inversion algorithms used in GPR work. This will include a short discussion of ill-posed problems utilizing a one dimensional model equation, a review of difficulties characteristic of tomographic inversion, a summary of results based on linearized inversion methods such as Devaney's generalized projection slice theorem, and finally, a brief overview of the some numerical algorithms. *iv.)* A discussion of a recent field study using GPR to image shallow targets.

1. FUNDAMENTAL EQUATIONS

The electromagnetic field vectors

$$\mathcal{E} = E(x)\exp(-i\omega t), \text{ and } \mathcal{H} = H(x)\exp(-i\omega t),$$

in an inhomogeneous medium with conductivity σ, permittivity ϵ and permeability μ (each of which may, in general, depend on x and ω) satisfy Maxwell's equations

$$\nabla \times E - i\omega\mu H = 0 \quad (1)$$
$$\nabla \times H + i\omega(\epsilon + i\sigma/\omega)E = 0 \quad (2)$$

We will be concerned with problems in which the parameters ϵ, σ, μ depend on position in the interior of a bounded domain D, the scattering region, and have constant values ϵ_0, σ_0, μ_0 in the exterior, $R^3 \setminus D$, of D. (For geophysical problems, it is more realistic to take the ground-air interface as $x_2 = 0$, let $R^3_\pm = \{x : \pm x_2 > 0\}$, assume $D \in R^3_-$ and that for $x \in R^3_\pm \setminus D$, $\epsilon = \epsilon_{0\pm}$, etc.. To simplify our presentation of the basic formulas, we will omit this additional complication, but will account for it in later discussions.) Far from the region D we impose the Sommerfeld condition of outgoing radiation. We assume that the scattering region is probed with a known (incident) wave E^i, H^i which satisfies Maxwell's equations in the background medium and is outgoing as $|x| \to \infty$. We write the total fields as a superposition, $E = E^i + E^s$, $H = H^i + H^s$, of incident plus scattered fields. It follows from Eqs.(1) and (2) that E satisfies

$$\mu\nabla \times \mu^{-1}\nabla \times E - k^2E = 0 \quad (3)$$

where $k^2 = \omega^2\mu(\epsilon + i\sigma/\omega)$. For the remainder of this paper we will assume that the magnetic permeability of D is the same as that of the background, $\mu(x,\omega) = \mu_0$ (or $\mu = \mu_{0\pm}$). With this approximation, which seems to be reasonable for most geophysical problems, Eq.(3) becomes $(\nabla \times \nabla \times -k^2I)E = 0$, and the incident field E^i satisfies this equation with $k = k_0$. Thus, the scattered field, E^s, is the solution of

$$\mathcal{L}_0E^s - (k^2 - k_0^2)E = 0 \quad (4)$$

where $\mathcal{L}_0 = \nabla \times \nabla \times -k_0^2I$. The dyadic Green's function for the operator $\mathcal{L}_0$ appearing in Eq.(4) is well known so we can write this equation in the form of an integral equation

$$E = E^i + \mathcal{L}_0^{-1}\left(k_0^2(n-1)E\right), \quad (5)$$

where $n(x,\omega) = k^2/k_0^2$ is the complex index of refraction. This is the LS integral equation which, in the direct problem determines the field E which results from the irradiation of the scattering region D by the given incident field E^i. For our discussion of the inverse problem, it is convenient not only to view E^i as a given vector field, but also to assume that E^s can (theoretically, at least) be measured over a surface S which

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surrounds or partially surrounds the scattering region D, and to consider Eq.(5) as a relation from which we can attempt to determine the complex refractive index, $n(x,\omega)$, of the scattering region. A rigorous presentation of the mathematics of inverse scattering including questions of uniqueness can be found in Colton and Kress[1], while physical and some computational aspects are dealt with in Chew[2].

2. APPROXIMATIONS

For the majority of GPR remote sensing applications in geophysics, the surface S over which the scattered field can be measured is a plane parallel to, or coincident with the ground-air interface. We shall assume this to be the case in the current presentation and take $S = \{x : x_2 = b\}$ (or a portion thereof), $b \geq 0$. We suppose that D lies in the region $R^3_- = \{x : x_2 < 0\}$, and that the Green's dyadic is constructed so that the appropriate conditions (continuity of normal H and tangential E) on $x_2 = 0$ are satisfied. (In other applications of inverse scattering methods, e.g. medical ultrasound tomography, the surface over which the scattered field is measured may completely enclose the domain D and so that additional data is available for the inversion process.) We observe first of all that the relation between n and E is a nonlinear one. This can be seen in a formal way by iterating Eq. (5) to produce the so-called Born series $E = E^i + \mathcal{L}_0^{-1}OE^i + \mathcal{L}_0^{-1}O\mathcal{L}_0^{-1}OE^i + \dots$, where $O = k_0^2(n-1)$. Each term in this series can be associated with increasingly complex multiple scatterings within the domain D. The integral equation obtained by truncating this series at the linear term is called the Born approximation. Of course, a solution of this equation neglects all multiple scattering terms and so is most accurate in the weak scattering limit. We obtain

$$E^s = k_0^2\mathcal{L}_0^{-1}(n-1)E^i = \int_D G_0(\cdot,y)O(y)E^i(y)\,dy \quad (6)$$

where G_0 is the dyadic Green's function. It will be noted that this integral equation, although linear, is of the first kind so some care must be taken in its inversion as we will discuss in the next section.

Another source of complexity is the vector nature of the basic integral equation, Eq. (5). In many problems of practical importance, it may be assumed that the variation of the properties of the scattering region D in one direction—which we may arbitrarily choose to be the x_3 direction—is negligible. In this case, the problem becomes two dimensional in the sense that all fields depend on x_1, x_2 only, and, as is well known, the vector nature of the radiation no longer needs to be considered. If we presume that the antenna launching the incident wave is oriented such that the electric field is in the x_3 (transverse) direction ($E = (0,0,E_3(x_1,x_2))$), then it is clear by symmetry that the scattered field will also lie in this direction. In this case, $E_3 \equiv u$ is determined as the solution of (recall μ is constant)

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)u + k^2u = 0$$

where $k = \omega^2\mu(\epsilon + i\sigma/\omega)$ is as defined above. The boundary conditions reduce to continuity of u and its normal derivative $\partial u/\partial x_2$ on the interface, and outgoing radiation at infinity. The LS equations simplifies accordingly and if it is assumed that the scattered field is measured on the interface, the integral equation

$$u^s(x_1,0) = \int_D G_0(x_1,\xi_1,\xi_2)O(\xi_1,\xi_2)u(\xi_1,\xi_2)\,d\xi_1 d\xi_2$$

where G_0 is now the Green's function for the scalar Helmholtz equation. Of course, for weak scattering this equation may be linearized as in the vector case.

3. INVERSION ALGORITHMS

As we have remarked above, the LS equation represents a nonlinear relation between the scattered field and the refractive index of the medium. Direct inversion of this equation clearly must involve numerical computation. We will only deal briefly with the full problem, and then concentrate on methods which apply in the linearized or weak scattering limit.

Suppose that the electric field is measured at points $\{p_j\}_{j=1}^{N_1}$ on the plane surface S, and that several values of incident field, $\{E_k^i\}_{k=1}^{N_2}$ are used, then we know that $E(x;E_k^i) = E(x,k)$ must satisfy Eq. (5) in R^3 and

$$E^s(p_j;E_k^i) = (\mathcal{L}_0^{-1}OE(\cdot,k))(p_j),\ p_j \in S$$

on the measurement surface. Perhaps the most direct approach to the solution of this pair of equations is via optimization techniques. A trial function pair $\tilde{E}(x,k)$, $\tilde{O}(x,k)$ is sought which minimizes the error

$$\begin{aligned}\epsilon \;&\equiv \textstyle\sum_{j,k} ||\tilde{E}(p_j,k) - E^s(p_j;E_k^i)||^2_{\text{surf}} + \\ &||\tilde{E}^s - (\mathcal{L}_0^{-1}\tilde{O}(\cdot,k)\tilde{E}(\cdot,k))||^2_{\text{vol}}\end{aligned}$$

where $||\cdot||_{\text{surf}}$ and $||\cdot||_{\text{vol}}$ are suitable norms introduced in the space of trial functions. For example, the problem is discreetized by introducing a finite set of node points in $\tilde{D} \supseteq D$ and S, expanding $\tilde{E}$, $\tilde{O}$ in terms a system of finite element basis functions based on these nodes, and using L^2 norms in the definition of the error, then after integration ϵ will become an ordinary function of the nodal values (and of the other parameters in the problem) and standard optimization procedures can be applied to it.

In the case of a homogeneous background medium, the solution of the linearized Born approximation integral equation may be found using generalized projection slice theorem (GPST) due to Devaney[3]. The inversion, in this case, is accomplished most easily for the source and receiver in the far field of region D so that the incident wave and the Green's function can both be simplified. The formulas which result relate the low pass filtered Fourier transform of the object function O to the measured scattered field. In the two-dimensional (2D), far field GPST case, $E_3^s = u^s(x_1, x_2)$ we obtain

$$u^s(x^+, x^-) = \frac{ie^{ik_0(\nu^+\cdot p^+ + \nu^-\cdot p^-)}}{8\pi k_0 d^+ d^-}\hat{O}(k_0(\nu^+ + \nu^-)),$$

where x_D is the centroid of D, $x^\pm = (l^\pm, 0)$ are the receiver and source locations respectively, $d^\pm = |x^\pm - x_D|$, $\nu^\pm = (x^\pm - x_D)/d^\pm$ are unit vectors, and $\hat{O}(\kappa) = \int \exp(-i\kappa\cdot x)O(x)\,dx$ is the two dimensional Fourier transform of the object function. Since we can only locate the transmitter and receiver on the surface S (line in 2D), then for fixed x^-, say, the object function is determined by the above formula only for wave vectors $\kappa = (\kappa_x, \kappa_y)$ which lie on a semi-circle of radius k_0 with center at $(k_0\nu_x^-, k_0\nu_y^-)$. Since $\nu_y^\pm > 0$ in this geometry, these semi-circles do not sweep out all of wave number space, but only a portion of the region $\{\kappa : \kappa_y > 0, |\kappa| < 2k_0\}$. Thus, regardless of the value of k_0, a unique determination of the object profile is not possible—only a filtered image can be obtained. Similar but more complex formulas can obtained using only the assumption that the depth of D is much greater than a wave length[4, 5]. For example, in the 2D situation described above with the transmitting and receiving antennas coincident monopoles, we obtain[4]

$$\hat{u}^s(k_1) = \frac{i\exp(i\pi/4)}{4k_0\sqrt{2\pi}\sqrt[4]{4k_0^2 - k_1^2}}\hat{O}\left(k_1, \sqrt{4k_0^2 - k_1^2}\right) \quad (7)$$

where $\hat{u}^s$ is the 1D Fourier transform of the scattered field measured along the horizontal line $x_2 = 0$. This result suffers from the same limitation on wave number space coverage that applies to the far field case. For purposes of later discussion we call the two results just presented the far field (FF) and Fourier transform (FT) approximations.

To my knowledge, the complete linearized problem including vector waves and refraction at the ground-air interface has yet to be solved (and would most certainly involve extensive numerical work). One major difficulty in inversion of the linearized problem stems from the fact that the relevant integral equation is of the first kind and has a very smooth kernel. This type of equation is severely ill-posed so that small errors in the measured data will cause large distortion of the predicted object function. Since such errors are bound to occur in a field application, some way of regularization must be used in the inversion process. For example, the optimization mentioned in the discussion of the numerical formulation of the nonlinear problem when applied in the linear case will clearly lead to a system of linear algebraic equations, and it will be found that the coefficient matrix of this system is ill-conditioned. Thus, some method such as singular value decomposition must be used in order to obtain the solution. As is well known, there are a number of possible ways to regularize in addition to singular value decomposition, and I will discuss very briefly the popular Tikhonov procedure[6] as applied to a prototype integral equation of the first kind which arose in connection with another tomographic problem[7].

The example that we will present is essentially that of a finite Laplace tranform which for convenience we assume applied to functions defined on the interval $[0,1]$. Thus, we consider the integral equation of the first kind $g(x) = \int_0^1 \exp(-xy)f(y)\,dy$, or $g = \mathcal{L}f$, where $g : [0,1] \to R^1$ is a given element of $L = L^2(0,1)$, and we assume that the solution is sought in the (Sobolev) space $H = H_0^1(0,1)$ of functions such that $f,\ f' \in L^2(0,1)$ and $f(0) = f(1) = 0$. Observe that the kernel of this equation is a very smooth, in fact, analytic function. Thus, we expect that inversion of $\mathcal{L}f = g$ may be an ill-posed problem. Multiplying by the adjoint of $\mathcal{L}$, see that any solution of the original problem satisfies $\mathcal{L}^*\mathcal{L}f = \mathcal{L}^*g$ so that for small α the solution of the regularized equation

$$\alpha f + \mathcal{L} * \mathcal{L}f = \mathcal{L} * g$$

should approximate the actual solution. The weak form of this equation may be solve by Galerkin's method, i.e., we seek $\hat{f}$ such that

$$\alpha(\hat{f}, h)_H + (\mathcal{L}^*\mathcal{L}\hat{f}, h)_H = (\mathcal{L}^*g, h)_H$$

for all functions $h \in H$. This equation may now be solved using standard finite element techniques. Efficient procedures exist for optimal selection of the regularization parameter[8].

4. FIELD STUDIES

We will now review two recent field studies[5] using GPR imaging. Both studies were done at sites located near the Oak Ridge National Laboratory: the pipeline and crypt sites. As the name implies, the pipeline site is a natural gas pipeline right of way, and the objective at this location was to image the buried pipe which represents an excellent approximation to a 2D object. Data was taken at 119 positions separated by 30 cm intervals along a line on the ground surface running perpendicular to the known direction of the pipe. The crypt site is a cemetery located within the Laboratory reservation and containing marked graves. Data was acquired at this site along a line on the ground surface and crossing two grave sites in a direction normal to the assumed orientation of the coffins. Measurements were made at 85 locations on the line separated by a distance of 15 cm. The geometry here was of course not strictly 2D, but the same imaging algorithms were employed as at the pipeline site. The soil at the sites had properties typical of moist clay, $n = n_1 + in_2$, with: $n_1 \approx 25$, $n_2 \approx 10^{-4}$. The same equipment was used in each study: a Sensors & Software Inc. pulse EKKO IV radar system with two 100 MHz nominal center frequency dipole antennas for sending and receiving. (The antenna calibration was done in air. At the sites, the center frequency was found to be approximately 60 MHz due probably to the conductivity of the soil.) The scattering data gathered by the instrument was treated using the 2D versions of the FF and FT approximations, and the log likelihood method. I will only discuss the results of the FT approach.

Equation (7) was used as the basis for processing the data obtained from both sites. As mentioned above, the FT inversion process is non unique. When employed at wave number k_0 using (2D) data, it filters out wave numbers above $2k_0$ from the image, and due to the restriction that the transmitter and receiver must both be located above or on the ground-air interface it does not cover the entire portion of the $2k_0$ radius disk. In addition to this basic filtering limitation, the following approximations are incorporated in Eq. (7): (*i*) 3D effects are neglected. These were not thought to be too important at either site, (*ii*) the transmitter and receiver are both assumed to lie in the same homogeneous medium as the scattering volume, i.e., ground-air interface effects are neglected. The received data clearly indicate the presence of large scale horizontal features, and these had to be filtered out prior to processing, (*iii*) the transmitting and receiving antennas are modeled as monopoles whereas the actual instrument used dipoles. The apparent effects of this neglect were again evidenced in the basic data as deviations between expected and actual signal amplitude behavior. Again, filtering of the data was used to reduce these effects.

Both items (*ii*) and (*iii*) above can always be at least partially accounted for by more realistic modeling of the source, receiver, and geometry of the actual experimental situation. On the other hand, some filtering of the data will always be necessary in order to smooth out unavoidable errors which arise from inadequate specification of the actual instrument configuration. In spite of the approximations inherent in our treatment of the data, the processing of the images obtained at both sites resulted in a clear indication of the location of the buried objects[1]. On the other hand, the images obtained lacked detail. Whether improved detail can be obtained by more careful modeling needs to be further investigated. The limitation on wave number space coverage which is imposed by the assumed GPR geometry acts as a bound on the amount of detail which can reasonably be expected. It is necessary, therefore, to acheive a reasonable compromise between the modeling effort required to achieve greater object resolution and the bounds on resolution imposed by the geometry.

5. REFERENCES

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[1]At the time of writing, figures illustrating the imaged objects are not available. These will be presented at the conference