PERTURBATION EFFECTS ON FILTERS HAVING LOCALISED POLES

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ABSTRACT

It is well known that the actual poles implemented in digital filters are increasingly sensitive to decreasing pole separation and that practical filters are implemented usually as a cascade of second order sections or the dual form to avoid this sensitivity. This paper considers the case of filters implemented in direct form and having localised poles. Expressions that describe pole position sensitivity for simple isolated poles and for multiple poles have previously been developed. This paper addresses the final problem of pole sensitivity due to pole interaction when one or more poles are placed in "close proximity" to other poles.

1. INTRODUCTION

The pole position sensitivity problem arising in the implementation of digital filters has been discussed at length and as a practical rule of thumb, high order filters are usually implemented as a cascade of second order sections. Cascaded low-order sections are used since extreme sensitivity arises when the poles are located close together [1,3]. Almost without exception, the common textbooks describing pole-zero sensitivity start their analysis with a calculus expression that assumes that the resulting expression will be differentiable [5,6]. In that case, the analysis for the simple pole sensitivity to parameter perturbation depends on a first order analysis that can be developed using a Taylor series expansion. Allen [1] has shown that it is improper to use this expression for the multiple pole case since the Taylor series first order term becomes zero and that same term becomes a divisor in the sensitivity expression. This paper explains the relationship between the simple and multiple pole cases.

The consideration of pole locations subject to coefficient perturbation is described by Marden [4]. The key results are that the locations of polynomial roots are continuous when the perturbation is applied. This is in contrast to the infinitesimal sensitivity theory Equation 1 which suggests that when the poles are multiple, the pole sensitivity is infinite in the sense that the pole position perturbation is also infinite. Kaiser [3] showed, for filters with simple transfer function poles only, that the perturbation of the pole p_l due to denominator polynomial coefficient a_k is given by the expression:

$$\Delta z = \frac{\sum_{k=1}^{N} \Delta a_k \, p_m^{N-k}}{\prod_{l=1,\, l\neq m}^{N} (p_m - p_l)} \qquad (1)$$

For transfer functions having multiple poles, Allen [1] has shown that the perturbation, Δz, of a pole having a multiplicity of M is given by the expression:

$$\Delta z = \left[\frac{\sum_{k=1}^{N} \Delta a_k \, p_m^{N-k}}{\prod_{l=1,\, p_l\neq p_m}^{N} (p_m - p_l)} \right]^{\frac{1}{M}} \qquad (2)$$

This result suggests that for multiple poles of order M, the pole position can be displaced as much as the 1/Mth power of the coefficient perturbation, Δa_k. Perhaps more importantly, since the pole perturbation is proportional to the Mth root of the right hand side of Equation 2, the pole locations will depend on the sign of the coefficient perturbation Δa_k as well as the separation of the remaining poles.

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The pole sensitivity for systems with simple and multiple poles has been described above, however a problem exists when poles are placed with decreasing separation. Equation 1 states the pole perturbation will increase to infinity, in the limit when the poles coincide, whereas Equation 2 provides a limit to pole perturbation for multiple poles. This interaction is the focus of the paper and will be examined in the following section.

2. Pole Sensitivity for Localised Poles

The relationship between the realised pole separation, Δp, and pole perturbation, Δz, will be examined by analysing the procedures used in generating Equation 1 and Equation 2.

The procedure of calculating pole sensitivity requires that a parameter of the realisation be perturbed and the resulting movement of the pole location be computed using a truncated Taylor series expansion. Consider a denominator polynomial D(z) having zeros, the poles of the system, denoted by p_l. $D^*(z)$ is derived from D(z) by perturbing coefficients a_k and can determined from the taylor series in Equation 3.

$$D^*(z_*) = D(z_*) - \sum_{k=1}^{N} \Delta a_k \, z_*^{N-k}$$
$$D(z+\Delta z) = D(z) + \frac{\Delta z}{1!}\frac{\partial D(z)}{\partial z}\bigg|_z + \dots + \frac{(\Delta z)^N}{N!}\frac{\partial^N D(z)}{\partial z^N}\bigg|_z \quad (3)$$

When Equation 3 is evaluated at a zero of $D^*(z)$, p_*, in the vicinity of a zero of D(z), p_m such that $p_*=p_m+\Delta z$, it can be combined to form Equation 4. The following equation forms the basis for the pole perturbation Equations, 1 and 2.

$$\sum_{k=1}^{N} \Delta a_k p_m^{N-k} \approx \frac{\Delta z}{1!}\frac{\partial D(z)}{\partial z}\bigg|_{z=p_m} + \dots + \frac{(\Delta z)^M}{M!}\frac{\partial^M D(z)}{\partial z^M}\bigg|_{z=p_m} + \dots \qquad M \le N \quad (4)$$

The perturbation, Δz, of a zero location of D(z), p_l, required to make $D^*(z)$ zero, is related to the parameter perturbation, Δa_k, as described in Equation 4. Simplification requires the identification of the dominant Taylor series term. It can be shown that for simple poles and multiple poles of order M this can be simplified to Equation 1 and Equation 2 respectively.

Assuming there are M-1 localised poles separated from p_m by a distance Δp, it can be shown that the dominant Taylor series term is either the first (for "large" Δp) or the Mth (for "small" Δp). By evaluating the separate Taylor series terms, the value of Δp for which the Mth Taylor series term becomes larger can be approximated with Equation 5.

$$\Delta p = \left| \frac{\sum_{k=1}^{N} \Delta a_k \, p_m^{N-k}}{\prod_{\substack{l=1 \\ \{\Delta p < |p_m - p_l|\}}}^{N} (p_m - p_l)} \right|^{\frac{1}{M}} \quad (5)$$

Equation 5 states that M poles cannot be placed inside a region whose radius defined by Equation 2, assuming that the M-1 closest poles to p_m form an Mth order pole at p_m. The approximated order of the pole, M, determines the rate at which the transition occurs, due to the Mth root in Equation 2. The following section provides examples that focus on the transition from the simple pole description of Equation 1 to the multiple pole case of Equation 2.

3. Results

Figure 1 and Figure 2 show the pole perturbation of a pole at 0.75 when surrounded by 4 poles separated by a distance of Δp. Figure 1 shows the location of the displaced poles as Δp is decreased and the polynomial

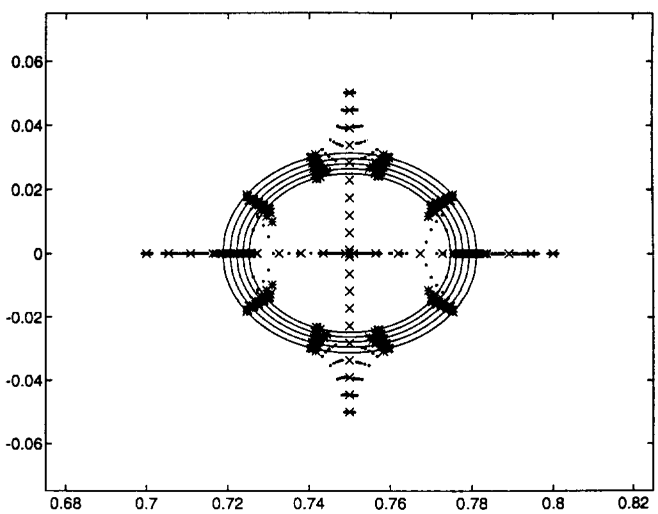

Figure 1 : Location of displaced poles due to coefficient perturbation and pole separation. "x" realised poles, "." & "*" displaced poles. The solid lines represent the regions whose radii are defined by Equation 2 and 5.

coefficients are perturbed individually by $\pm 2^{-25}$. MATLAB is used in the analysis to locate the resulting pole displacement. The realised poles are indicated as 'x' and the perturbed poles are displayed as '.', for the case when the realised pole is outside the region defined by Equation 5, and '*', when inside. Figure 2 shows the displacement of the pole at 0.75, as the pole separation is reduced, for each perturbation of the polynomial coefficient. The vertical lines depicts the pole separation defined by Equation 5 and shows the location of the distinct change in pole displacement due to transition of dominant Taylor series terms. Since the pole displacement is calculated as the Mth root of a function, the rate of increase is related to the order of the pole, M.

Figure 3 depicts a more general case where a pole is located at 0.75 and surrounded by (M-1) poles, where M is varied from 2 to 10, the pole separation is varied from 0.2 to 0.001 and a_M is perturbed by $\pm 2^{-25}$. The resulting pole displacement is normalised by the displacement described in Equation 2 and graphed against system order and pole separation. The vertical partition indicates the boundary defined by Equation 5. This figure shows the rapid change in pole displacement as M poles occupy a region whose size is defined by Equation 5 for the range of pole order.

Figure 4 and Figure 5 show the pole displacement of a second order pole at 0.75 when a pair of complex conjugate poles are brought within close proximity. Again each polynomial coefficient is perturbed by $\pm 2^{-25}$. Figure 5 shows the initial displacement to be offset by a value calculated using Equation 2 for a second order pole. The pole displacement then increases indicating a pole multiplicity of 4 as the complex conjugate pole approach closer than the distance estimated by Equation 5 setting M=4.

The significant Taylor series term changes dramatically as pole separation reduces resulting in the pole position perturbation definition switching from Equation 1 to Equation 2. This transition occurs when M poles attempt to occupy a region whose size is defined by Equation 5, assuming the M poles to be a multiple pole of order M. Developing from this, it is shown that for a prescribed coefficient perturbation the system poles cannot approach closer than a distance approximated by the multiple pole Equation 2. Once the poles move inside they are forced into a positions around the circle defined by the multiple pole equation.

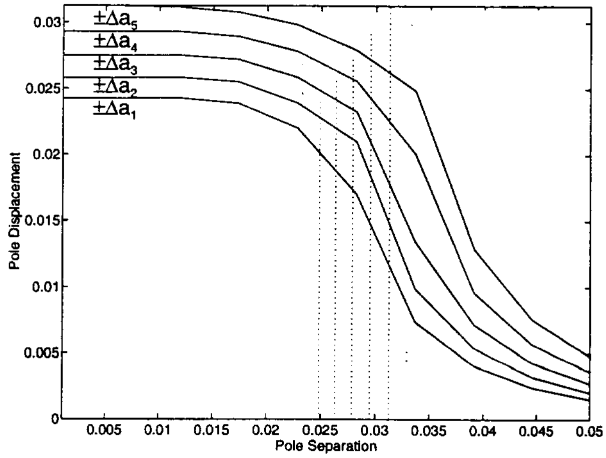

Figure 2 : Displacement of the pole at 0.75, due to individual coefficient perturbation, as a function of pole separation. The dotted lines represent the pole separation defined by Equation 5.

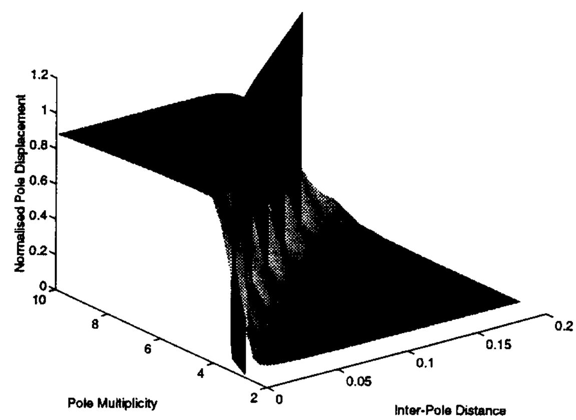

Figure 3 : Normalised pole displacement as a function of pole order and inter-pole distance.

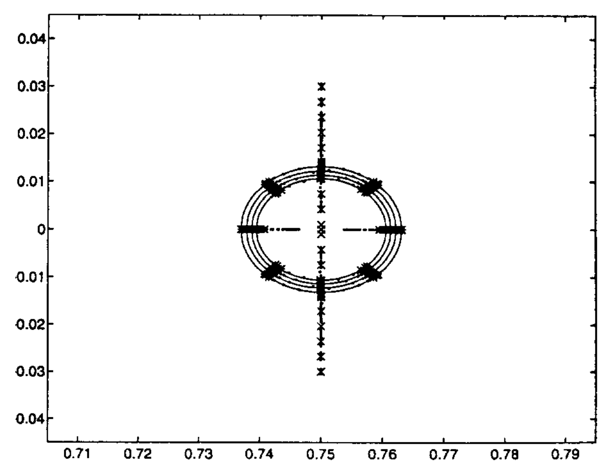

Figure 4 : Location of displaced poles due to coefficient perturbation and pole separation. The solid lines represent the regions whose radii are defined by Equation 2 and 5.

4. CONCLUSIONS

In this paper we have considered the effect of coefficient perturbation for system functions having multiple poles. The effects of realistic sized coefficient perturbations on pole positions are found to be described by the multiple pole perturbation equation for isolated multiple poles. For multiple poles near to simple poles, the multiple perturbation equation should be used when the simple poles are inside the perturbation circle computed from the multiple pole perturbation equation. In effect, the simple pole close to the multiple pole becomes contained in the multiple once the coefficient perturbation exceeds a simply computed value. We thus can account satisfactorily for the perturbation of a mixture of simple poles and multiple poles subject to coefficient perturbation and avoid the infinite sensitivity prediction that arises from the infinitesimal sensitivity analysis.

5. REFERENCES

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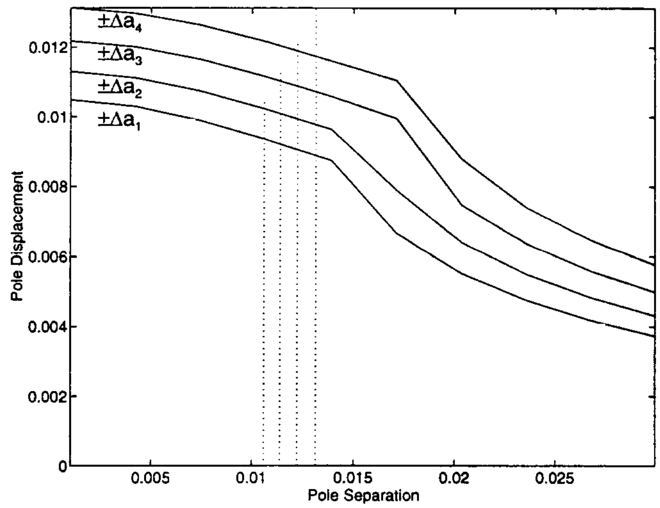

Figure 5 : Displacement of the pole at 0.75, due to individual coefficient perturbation, as a function of pole separation. The dotted lines represent the pole separation defined by Equation 5.