H_∞ DECONVOLUTION FILTER DESIGN AND ITS APPLICATION IN IMAGE RESTORATION

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ABSTRACT

This paper addresses the use of H_∞ filtering to deconvolution, in particular, to the problem of image restoration. The proposed H_∞ deconvolution filter has some advantages in the image restoration such as it can deal with unknown boundary problem and spatially varying blurs. In this paper, H_∞ filter is compared with the inverse Wiener filter and a regularized restoration. The experimental results show that the H_∞ filter deals with the unknown boundary problem better than the Wiener filter. Compared with the regularization method, it gives a sharper restored image, especially, when the original image contains many details.

1. INTRODUCTION

H_∞ filtering has attracted a great deal of attention in recent years. Recently, progress has been made in the design of H_∞ filters [1], [2], [3] and also in their practical applications [4]. Compared with conventional stochastic filters, H_∞ filters have some practical advantages. The H_∞ filtering does not require statistical information about noise, instead, the only requirement is that the noise has bounded energy. Therefore, it is more robust to the noise uncertainty and also less sensitive to the uncertainty in the signal dynamics.

The objective of an H_∞ filter is to guarantee the H_∞ norm of the operator from the external disturbance to the estimate error below a prescribed bound. For an H_∞ deconvolution filter, we can consider the uncertainty of the input signal to a system and observation noise as the external disturbances. The deconvolution filters proposed in [1] and [2] are for both time invariant, discrete-time and continuous-time systems, respectively. The former is obtained by a polynomial approach. This paper gives an H_∞ filter for discrete-time and time varying systems in state space. It is obtained by an algebraic approach.

Since the usual causes of image degradation are blur and additive noise, the specific objectives for image restoration are to deblur and simultaneously attenuate noise. An image has finite energy since it has a finite size and the intensity is typically limited to the range 0-255. Therefore, we can apply the H_∞ filter to image restoration.

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The paper is organized as follows. First, we present an H_∞ deconvolution filter. Next, it is compared with the inverse Wiener filter and regularization method in a practical image restoration example. The paper concludes with a brief summary of advantages of the H_∞ formulation.

Throughout the paper we shall use the following notations: R^n denotes the n dimensional vector space; $(.)^T$ denotes the transpose of a matrix; I_n denotes an identity matrix with dimension n; $L_2[0, N-1]$ denotes the square summable function space, e.g.,

$$\begin{aligned} L_2[0,N-1] &= \{f : \|f\|_2 < \infty\} \\ \|f\|_2 &= \left(\sum_{k=0}^{N-1} f^T(k)f(k)\right)^{1/2} \end{aligned}$$

2. H_∞ DECONVOLUTION FILTER DESIGN

Consider a discrete-time, time varying system as follows,

$$\begin{aligned} x(k+1) &= A(k)x(k) + B_1(k)u(k) + B_2(k)w(k), x(0) = x_0 \\ y(k) &= C(k)x(k) + D_1(k)u(k) + D_2(k)w(k) \end{aligned} \quad (1)$$

where $k \in [0, N-1]$ is an integer index; $x \in R^n$ is the state; $u \in R^m$ is the input; $y \in R^p$ is the measurement, $w \in R^q$ is the noise, and $x_0 \in R^n$ is the initial state of the system. The matrices $A(k)$, $B_1(k)$, $B_2(k)$, $C(k)$,$D_1(k)$ and $D_2(k)$ are bounded functions of k with proper dimensions.

We assume $D_1(k) \neq 0$ for all $k \in [0, N-1]$. And suppose that the input u consists of two parts:

$$u(k) = u_0(k) + u_u(k) \quad (2)$$

where u_0 is the known part and u_u is the unknown part. We also assume that u_u and w have finite energy over the interval $[0, N-1]$, e.g., u_u, $w \in L_2[0, N-1]$.

A general, n-th order causal deconvolution filter may be expressed as follows,

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}(k)\hat{x}(k) + \hat{B}_1(k)u_0(k) + \hat{B}_2(k)y(k) \\ \hat{u}(k) &= \hat{C}(k)\hat{x}(k) + \hat{D}_1(k)u_0(k) + \hat{D}_2(k)y(k) \end{aligned} \quad (3)$$

where $\hat{x} \in R^n$ is the filter state; $\hat{u} \in R^m$ is the filter's output which is the estimate of u; $\hat{A}(k)$, $\hat{B}_1(k)$, $\hat{B}_2(k)$, $\hat{C}(k)$, $\hat{D}_1(k)$ and $\hat{D}_2(k)$ are the filter parameters to be determined.

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The deconvolution filter should be designed to minimize the estimate error $e(k) = \hat{u}(k) - u(k)$. Therefore, the objectives for a suboptimal deconvolution filter should be as follows,

$$\sup_{u_u, w \in L_2[0,N-1], x_0 \in R^n} \frac{\|u - \hat{u}\|_2^2}{\|u_u\|_2^2 + \|w\|_2^2 + x_0^T S x_0} < \gamma^2 \quad (4)$$

for x_0 unknown or

$$\sup_{u_u, w \in L_2[0,N-1]} \frac{\|u - \hat{u}\|_2^2}{\|u_u\|_2^2 + \|w\|_2^2} < \gamma^2 \quad (5)$$

for x_0 known, where γ is a prespecified positive number and S is a positive-definite matrix .

We can prove that the suboptimal filter satisfying (4) or (5) should be as follows,

$$\begin{aligned}
\hat{A}(k) &= A(k) - \hat{B}_2(k)C(k) \\
\hat{B}_1(k) &= B_1(k) - \hat{B}_2(k)D_1(k) \\
\hat{B}_2(k) &= (A(k)Q(k)C^T(k) + B(k)D^T(k))R^{-1}(k) \\
\hat{C}(k) &= -D_1^T(k)R^{-1}(k)C(k) \qquad (6) \\
\hat{D}_1(k) &= I - D_1^T(k)R^{-1}(k)D_1(k) \\
\hat{D}_2(k) &= D_1^T(k)R^{-1}(k)
\end{aligned}$$

where

$$\begin{aligned}
B(k) &\equiv [B_1(k) \quad B_2(k)] \\
D(k) &\equiv [D_1(k) \quad D_2(k)] \\
R(k) &= C(k)Q(k)C^T(k) + D(k)D^T(k)
\end{aligned}$$

if there exists a symmetric solution $Q(k)$ to the following Riccati equation:

$$\begin{aligned}
Q(k+1) = {} & A(k)Q(k)A^T(k) + B(k)T_1^{-1}B^T(k) \\
& -(A(k)Q(k)C^T(k) + B(k)T_1^{-1}D^T(k)) \qquad (7) \\
& T_2^{-1}(k)(C(k)Q(k)A^T(k) + D(k)T_1^{-1}B^T(k))
\end{aligned}$$

with $Q(0) = S^{-1}$ for objective (4) or $Q(0) = 0$ for objective (5), where T_1 and $T_2(k)$, $k \in [0, N-1]$, are both positive-definite matrices:

$$\begin{aligned}
T_1 &= (I_{m+q} - \gamma^{-2}L^T L) \\
T_2(k) &= D(k)T_1^{-1}D^T(k) + C(k)Q(k)C^T(k) \\
L &= [I_m \quad 0]
\end{aligned}$$

The proof is omitted since it is too lengthy to be contained in this paper.

Here we can see that γ has to be greater than 1 because the condition $T_1(k) > 0$ has to be satisfied. Based on the filter formulation described above, the deconvolution process can be repeated in order to improve the estimation iteratively. For the initial estimation, u_u may just be set as zero if we do not have any knowledge about the input signal. Then, for the latter iteration, we can use the previous estimate $\hat{u}$ as u_0 to improve the estimate based on the previous estimation.

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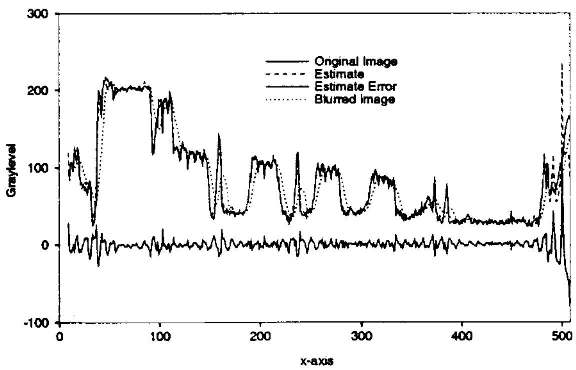

Figure 1: Restoration results by Wiener filter

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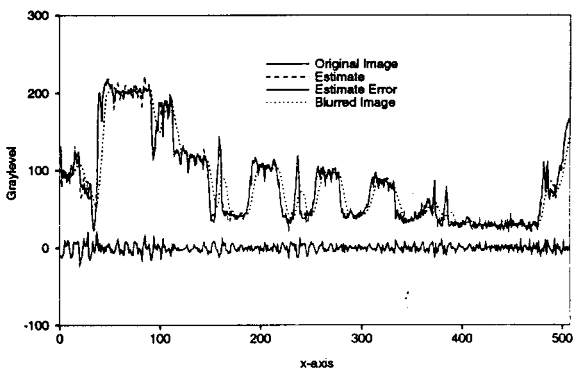

Figure 2: Restoration results by H_∞ filter

When the initial condition x_0 is known and there is no noise in the system, the inverse filter in time domain can be recovered from the H_∞ filter above by setting $B_2(k)$ and $D_2(k)$ to zero matrices if $D_1(k)$ is square and invertible for $k \in [0, N-1]$. In this case, the estimate $\hat{u}$ is exactly the same as the input signal u without any error.

3. PRELIMINARY RESULTS

For the comparison of Wiener filter and H_∞ filter, one row of an image with 500 pixels wide blurred by 9 taps horizontal uniform motion is used as a 1-D example. The estimate by Wiener filter is also 500 pixels wide. Figure 1 shows that there are ringing distortions near the boundaries, especially the right boundary, because of the unknown boundary problem as mentioned in [5].

To reduce the boundary effects, first, two causal state-space models are obtained from the blurring function for two space reference directions, from left to right (forward) and from right to left (backward). Zero is used as the boundary condition of the model for the forward direction. Based on the estimate obtained in the forward direction, we can get the boundary condition for the model in the backward direction. The boundary estimate for the forward model obtained from the backward estimation is of closer to the true boundary values than the initial guess, zero. Repeating the process, we can improve the estimate of the boundary. In this way, not only is the unknown boundary problem addressed but we can also estimate all the pixels of the original image. The results in Figure 2 demonstrate that the unknown boundary effect is eliminated and the estimation gives the greylevels of all 508 pixels that contributed to the blur.

We use the pepper image as an example of comparing H_∞ filter and regularization method. The degraded image shown in Figure 4 is one section with the size 256×256 of the 512×512 original image blurred by an 11 pixel uniform horizontal motion and added noise at 30dB BSNR. The 256×256 portion of the original image corresponding to the extracted block of the blurred image is shown in Figure 3.

The restored image by regularization method is shown in Figure 5. The image was restored by using a conjugate-gradients iterative technique to obtain a regularized restoration [6]. The problem of unknown boundary values was handled by restoring the boundary pixels along with the rest of the image. Space-invariant regularization was used with a Laplacian regularization operator and a regularization parameter chosen to minimize the MSE of the restoration. The Laplacian serves as an approximate whitening filter, which makes this restoration approximate a Wiener filter [7].

The image restored by H_∞ filter is shown in Figure 6. For the H_∞ deconvolution filter design, we need the state space models in two directions (from left to right and the converse) for the blurring function for the11 pixel horizontal uniform motion. Since the blurring function is symmetric over the horizontal axis, the models for two directions are the same:

$$A = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \end{pmatrix}$$

$$B_1 = (1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)^T$$

$$C = (1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1)/11$$

$$B_2 = \mathbf{0}_{10\times 1}, \quad D_1 = 1/11$$

D_2 is a free scalar parameter. We can choose its values according to the intensity of noise. D_2 is set as zero if there in no noise. Its value should be increased as the noise intensity going up. For the choice of matrix S, it should be selected as a small positive-definite matrix if we want to reduce the effect of the unknown boundary on the estimate. In this experiment, we chose D_2 and S as 0.15 and $0.01I_n$, respectively.

Figure 7 gives the mean square errors of the restored images restored by H_∞ filter and regularization method compared with the original image for each row. It shows that the regularization method works better than the H_∞ filter at the smooth area, especially at the lower part of the image. The latter restored the central part better where the image contains more details. The MSEs in Figure 7 also shows that the H_∞ filter performs consistently for the whole image which has low frequency components at top and lower rows and high frequency components at central rows. The experiment results show that the H_∞ filter works better for high detail images and conversely, the performance of regularization method is better for smooth images.

4. COMMENTS AND CONCLUSIONS

If the observed data sequence is infinite in extent, the effect of boundary condition decays to zero by using the H_∞ filter. However, when the data sequence is finite in extent as is the case in image processing, boundary effects can be quite significant. Applying the H_∞ filter to the same data sequence back and forth repeatedly can reduce the boundary effects significantly. The results above show that the H_∞ filter can outperform the Wiener filter in image restoration with unknown boundary conditions. Compared with regularization method, H_∞ filter works better for images with many details. It performs consistently for both smooth and rough sections of the image for the test. Furthermore the H_∞ filter does not need signal and noise models. Another advantage of the H_∞ filter is that it can be applied to the same data sequence repeatedly to improve the estimate accuracy by using the last estimate as u_k while its initial values can simply be set as zeros. Finally, this filter can be used for linear shift varying processing of images.

5. REFERENCES

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Figure 3: The original image 256 by 256

Figure 4: The blurred and noisy image

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Figure 5: The restored image by regularization method

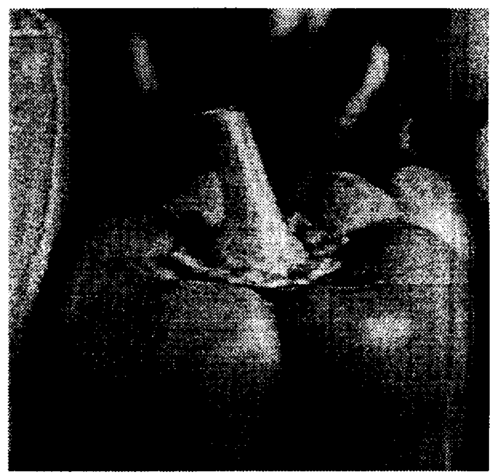

Figure 6: The restored image by H_∞ filter

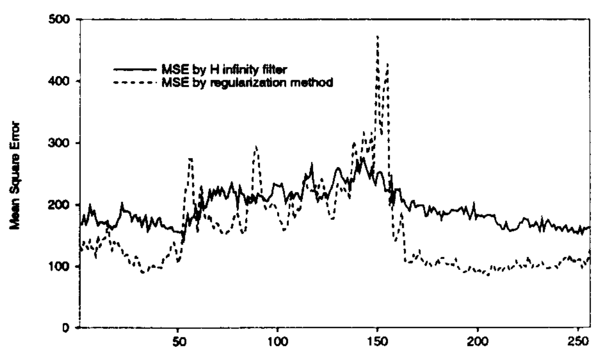

Figure 7: The performance comparison