FAST IMAGE SYNTHESIS USING AN EXTENDED SELF-SIMILAR (ESS) MODEL

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ABSTRACT

In this work, we propose a method called incremental Fourier synthesis to generate images based upon the 2-D extended self-similar (ESS) model. This algorithm creates the stationary increments of ESS processes by Fourier synthesis. Then, the increments are added up to generate the nonstationary 2-D ESS process. Because the new method can take advantage of the FFT, its computational complexity is only $O(N^2 \log_2(N))$, and its memory requirement is $O(N^2)$ for an image of size $N \times N$.

1. INTRODUCTION

Fractional Brownian motion (fBm) is a stochastic model which is useful to describe many natural phenomena [5]. In computer graphic applications, the generation of realizations of 2-D fBm is used to create natural looking landscapes and clouds [4], [6]. A major disadvantage of fBm is that the appearance of its realizations is controlled by a single parameter H known as the Hurst parameter. Even though a natural texture may exhibit similar roughness over a large range of scales, it is improper in reality to assume the roughness to be constant for arbitrary large or small scales. Recently, we introduced a 2-D extended self-similar (ESS) process to model landscapes that have varying degrees of roughness at different scales [2]. We demonstrated how parameters can control the size of bays or roughness of the coastline at various scales.

In this work, we introduce a fast new method called incremental Fourier synthesis to generate 2-D ESS where the stationary increments of ESS are created by a Fourier synthesis method. The increments are added up to generate the nonstationary 2-D ESS process. Since the new method takes advantage of the FFT, its computational complexity is only $O(N^2 \log_2(N))$, and its memory requirement is only $O(N^2)$. Moreover, the method uses statistics which are as close as possible to the exact ESS statistics.

This paper is organized as follows. Section 2 provides a brief overview of ESS processes. Section 3 presents some background material about stationary periodic random fields which is necessary to understand the synthesis algorithm presented in Section 4. The new algorithm is compared with standard Fourier synthesis for generating 2-D fBm in Section 5, and some concluding remarks are given in Section 6.

2. 2-D EXTENDED SELF-SIMILAR PROCESSES

To model 2-D landscapes, we define a 2-D ESS process $B_f(t_x, t_y)$ to be a mean zero multivariate Gaussian process such that at the origin $B_f(0,0) = 0$ and the variance of the increments of $B_f(t_x, t_y)$ follow a power law of

$$\mathrm{VAR}[B_f(t_x + r_x, t_y + r_y) - B_f(t_x, t_y)] = \sigma^2 f(r_x, r_y) \quad (1)$$

where $\sigma^2 = \mathrm{VAR}[B_f(s_x + 1, s_y) - B_f(s_x, s_y)]$ and

$$f(r_x, r_y) = \tilde{f}(\|(r_x, r_y)^T\|_{\mathbf{R}}).$$

Note that $\|\vec{t}\|_{\mathbf{R}} = \sqrt{\vec{t}^T \mathbf{R} \vec{t}}$ for a positive definite $\mathbf{R}$. Equation (1) is the 2-D extended self-similarity condition, and it means that the variance of any increments is dependent only on the length of the increment. The function $\tilde{f}(\cdot)$ is known as the structure function, and the form of the structure function is not completely arbitrary [1]. The structure function controls the appearance of 2-D ESS processes as discussed in [2], [3]. Note that when $\tilde{f}(s) = |s|^{2H}$, then the ESS processes is simply fBm. Thus, fBm forms a subset of ESS processes.

Two dimensional ESS processes are nonstationary. Their increments, however, form stationary processes. We define the first order discrete increments of 2-D ESS processes as,

$$I_x(m_x, m_y) = B_f(m_x + 1, m_y) - B_f(m_x, m_y),$$
$$I_y(m_x, m_y) = B_f(m_x, m_y + 1) - B_f(m_x, m_y).$$

The second order increments of 2-D ESS processes is defined as,

$$\begin{aligned} I_2(m_x, m_y) &= I_x(m_x, m_y + 1) - I_x(m_x, m_y) \\ &= I_y(m_x + 1, m_y) - I_y(m_x, m_y) \\ &= B_f(m_x + 1, m_y + 1) + B_f(m_x, m_y) \\ &\quad - B_f(m_x + 1, m_y) - B_f(m_x, m_y + 1). \end{aligned} \quad (2)$$

The first and second order increments are stationary, and the correlation functions of the increments are,

$$r_x(m_x, m_y) = \frac{\sigma^2}{2}[f(m_x + 1, m_y) + f(m_x - 1, m_y) - 2f(m_x, m_y)], \quad (3)$$

$$r_y(m_x, m_y) = \frac{\sigma^2}{2} f(m_x, m_y + 1) + f(m_x, m_y - 1) - 2f(m_x, m_y)], \quad (4)$$

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$$r_2(m_x,m_y) = \frac{\sigma^2}{2}[2(f(m_x+1,m_y)+f(m_x-1,m_y)$$
$$+f(m_x,m_y+1)+f(m_x,m_y-1)) - (f(m_x+1,m_y+1)$$
$$+f(m_x+1,m_y-1)+f(m_x-1,m_y+1)$$
$$+f(m_x-1,m_y-1)) - 4f(m_x,m_y)]. \quad (5)$$

3. PERIODIC RANDOM FIELDS

Consider a periodic stationary Gaussian random field whose correlation function satisfies

$$R(m_x+kN, m_y+lN) = R(m_x,m_y), \qquad \forall k,l \in \mathbf{Z}. \quad (6)$$

Each realization of this random field is also periodic with a period of N in both the x and y directions. Thus, it is only necessary to know the values of the field over an $N \times N$ lattice of points, and the correlation function only needs to be considered for time lags which lie on an $N \times N$ grid. Due to symmetry of the correlation function about the origin and (6), we have

$$R(m_x,m_y) = R(N-m_x, N-m_y) = R(m_x-N, m_y). \quad (7)$$

Equation (7) uniquely determines the correlation function on the lattice $m_x, m_y \in [0, N-1]\times[0,N-1]$ when given the values for lags $m_x \in [-N/2+1, N/2]$ and $m_y \in [0, N/2]$.

The importance of periodic random fields is due to the fact that the 2-D DFT is the Karhunen-Loève transform (KLT) for such fields. A nice result of this property is that realizations of periodic random fields are easy to generate because one just needs to scale white noise by the square-root of the field's power spectrum and then calculate the inverse 2-D DFT. In fact, this generation procedure is used in normal Fourier synthesis of fBm where the power spectrum is assumed to be

$$\hat{R}(k_x,k_y) = C/\sqrt{k_x^2+k_y^2}^{\,2H+2} \quad \text{for } k_x,k_y = 0,\ldots,N/2. \quad (8)$$

The other values of the power spectrum are determined by symmetrically expanding $\hat{R}(k_x,k_y)$. Usually, the first $N/2 \times N/2$ values of the generated field are taken as the fBm image in order to avoid artifacts from the periodicity of the field.

4. INCREMENTAL FOURIER SYNTHESIS

The idea to create samples of ESS over an $(M+1)\times(M+1)$ grid is to generate the stationary increments over an $M \times M$ grid. We attempt to create periodic random fields of size $N \times N$ (where $N = 2M$) whose correlation function $R(m_x,m_y)$ matches the correlation function of the nonperiodic increments for $(m_x,m_y) \in [-M,M]\times[-M,M]$. The other values of the correlation function for the periodic field can be determined via symmetries. Then the increments can be synthesized by using the corresponding power spectrum to scale white noise.

Before we describe the new synthesis algorithm, it is worthwhile to point out two issues. One problem to consider is that the target periodic correlation function may not be positive definite. Due to the Gibbs phenomenon, some of the values of the DFT of the desired periodic correlation function may be negative. By considering these bad values to be zero, we create the actual power spectrum which generates the increments. Another point to consider is that the 1st and 2nd order increments cannot be generated independently or else major creasing will appear. The dependence of $I_2(m_x,m_y)$, $I_x(m_x,m_y)$, and $I_y(m_x,m_y)$ is due to (2) where the subtraction is taken modulo N. Now, we describe the new algorithm in detail below.

Algorithm: Incremental Fourier Synthesis Method

1. Create white noise processes such that for $k_x = 0,\ldots,N$, and $k_y = 0,\ldots,N/2$, $\hat{W}(k_x,k_y) \sim N(0,1)$, $\hat{\phi}(k_x,k_y) \sim \text{Uniform}[0,2\pi)$, and $\phi(0,0) = \phi(N/2,N/2) = \phi(N/2,0) = \phi(0,N/2) = 0$.

2. Calculate $R_2(m_x,m_y)$ (the desired correlation function of $I_2(m_x,m_y)$) by (5) for $m_x = -N/2+1,\ldots,N/2$ and $m_y = 0,\ldots,N/2$,. Then symmetrically expand the correlation function via (7).

3. Calculate the power spectrum by

$$\hat{R}_2(k_x,k_y) = \text{FFT}_{m_x}[\text{FFT}_{m_y}[R_2(m_x,m_y)]].$$

4. Let the actual positive semidefinite power spectrum $\hat{S}(k_x,k_y) = 0$ when $k_x = 0$ or $k_y = 0$. Otherwise, $\hat{S}(k_x,k_y) = \max(0, \hat{R}(k_x,k_y))$.

5. Synthesize the DFT coefficients of $I_2(m_x,m_y)$ so that for $k_x = 0,\ldots,N-1$ and $k_y = 0,\ldots,N/2$:

$$\hat{I}_2(k_x,k_y) = N\sqrt{\hat{S}_2(k_x,k_y)}\hat{W}(k_x,k_y)e^{j\phi(k_x,k_y)},$$

and $\hat{I}_2(k_x,k_y) = \hat{I}_2^*(N-k_x, N-k_y)$ elsewhere.

6. Calculate the 2nd order increments for $m_x, m_y = 0,\ldots, M-1$:

$$I_2(m_x,m_y) = \text{IFFT}_{m_x}[\text{IFFT}_{m_y}[\hat{I}_2(k_x,k_y)]].$$

7. Create white noise processes such that for $k_x,k_y = 0,\ldots,N/2$, $\hat{W}_x(k_x) \sim N(0,1)$, $\hat{W}_y(k_y) \sim N(0,1)$, $\phi_x(k_x) \sim \text{Uniform}[0,2\pi)$, $\phi_y(k_y) \sim \text{Uniform}[0,2\pi)$, and $\phi_x(0) = \phi_y(0) = \phi_x(N/2) = \phi_y(N/2)$.

8. Calculate $R_x(k_x,k_y)$ and $R_y(k_x,k_y)$ for $k_x,k_y = 0,\ldots, N/2$ using (3) and (4). Symmetrically expand the correlation functions using (7).

9. Compute the desired power spectrum of the 1st order increments at the zero frequencies via

$$\hat{R}_x(k_x,0) = \text{FFT}_{m_x}[\textstyle\sum_{m_y=0}^{N-1} R_x(m_x,m_y)],$$
$$\hat{R}_y(0,k_y) = \text{FFT}_{m_y}[\textstyle\sum_{m_x=0}^{N-1} R_y(m_x,m_y)].$$

10. Define the actual positive semidefinite power spectrum of the 1st order increments at the zero frequencies via

$$\hat{S}_x(k_x,0) = \max(0, \hat{R}_x(k_x,0)),$$
$$\hat{S}_y(0,k_y) = \max(0, \hat{R}_y(0,k_y)).$$

11. Synthesize the DFT coefficients of the 1st order increments so that $\hat{I}_x(k_x,k_y)$ is equal to

$$\begin{cases} -j\frac{\hat{I}_2(k_x,k_y)}{2\sin(2\pi k_y/N)}e^{-j2\pi k_y/N}, & \text{for } k_x,k_y = 1,\ldots,N-1, \\ N\sqrt{\hat{S}_x(k_x,0)}\hat{W}_x(k_x)e^{j\phi_x(k_x)}, & \text{for } k_x = 0,\ldots,N/2, \text{ and } k_y = 0, \\ \hat{I}_x^*(N-k_x,0), & \text{for } k_x = N/2+1,\ldots, N-1, \text{ and } k_y = 0, \\ 0, & \text{otherwise.} \end{cases}$$

and $\hat{I}_y(k_x, k_y)$ is equal to

$$\begin{cases} -j\frac{\hat{I}_2(k_x,k_y)}{2\sin(2\pi k_x/N)}e^{-j2\pi k_x/N}, & \text{for } k_x, k_y = 1, \ldots, N-1, \\ N\sqrt{\hat{S}_y(0,k_y)}\hat{W}_y(k_y)e^{j\phi_y(k_y)}, & \text{for } k_x = 0 \text{ and } k_y = 0, \ldots, N/2, \\ \hat{I}_y^*(0, N-k_y), & \text{for } k_x = 0 \text{ and } k_y = N/2+1, \ldots, N-1, \\ 0, & \text{otherwise.} \end{cases}$$

12. Compute the 1st order increments along the image boundaries for $m_x, m_y = 0, \ldots, M-1$:

$$I_x(m_x, 0) = \frac{1}{N}\mathrm{IFFT}_{m_x}[\textstyle\sum_{m_y=0}^{N-1} \hat{I}_x(k_x, k_y)],$$
$$I_y(0, m_y) = \frac{1}{N}\mathrm{IFFT}_{m_y}[\textstyle\sum_{m_x=0}^{N-1} \hat{I}_x(k_x, k_y)].$$

13. Add up the increments to calculate the ESS field for $m_x, m_y = 0, \ldots, M$ via

$$\begin{aligned} &B_f(0,0) = 0, \\ &B_f(m_x, 0) = B_f(m_x-1, 0) + I_x(m_x-1, 0), \\ &B_f(0, m_y) = B_f(0, m_y-1) + I_y(0, m_y-1), \\ &B_f(m_x, m_y) = B_f(m_x, m_y-1) + B_f(m_x-1, m_y) \\ &-B_f(m_x-1, m_y-1) + I_2(m_x-1, m_y-1). \end{aligned}$$

5. COMPARISON TO FOURIER SYNTHESIS

In this section, we compare incremental Fourier synthesis to standard Fourier synthesis for the application of generating samples of 2-D fBm. First, we calculate the synthesized correlation functions of the two methods by computing the inverse FFT of the power spectrums that were used to scale the white noise. Then, we use the correlation function to calculate the structure function based on the x directed increments of the generated picture, i.e.

$$f(d) = \frac{\mathrm{VAR}[B_f(m_x+d, m_y) - B_f(m_x, m_y)]}{\mathrm{VAR}[B_f(m_x+1, m_y) - B_f(m_x, m_y)]}, \qquad d \in \mathbf{Z}^+.$$

To get a local measurement of the rate the structure function is increasing with respect to scale we define a generalized scale dependent Hurst parameter as,

$$\tilde{H}(m) = \frac{1}{2}\log_2(\tilde{f}(2^{m+1})/\tilde{f}(2^m)).$$

The values $\tilde{H}(m)$ for the incremental and standard method with H set to 0.2 and the image size set to 512×512 are shown in Figure 1. The figure shows that the actual process generated by incremental Fourier synthesis is nearly constant, i.e. virtually self-similar. It is also clear that images generated by standard Fourier synthesis are not statistically self-similar. In fact, the figure suggests that the generated images will be smoother at finer scales since the value of $\tilde{H}(m)$ becomes larger.

Figures 2 and 3 show the images generated by the standard and incremental Fourier methods, respectively, at different scales where $H = 0.2$. At each scale, the resolution of the picture is 64×64, and each picture is scaled so that the dynamic range of the pixel values cover all 64 gray level values. The statistical self-similarity is evident for the fBm realization created by our new method. As predicted by the generalized Hurst parameters, the fBm realization generated by traditional Fourier synthesis is smoother at finer scales.

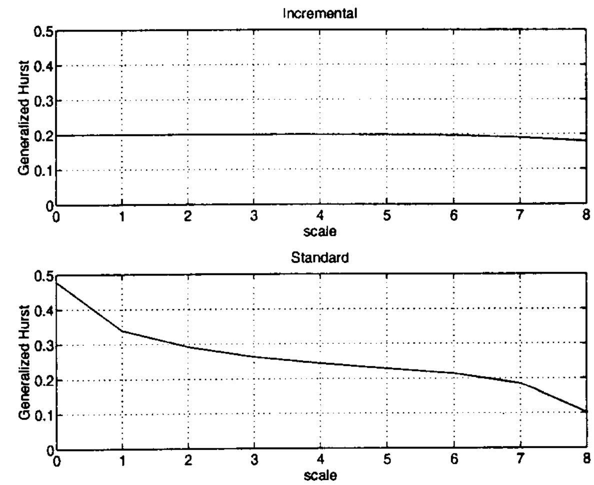

Figure 1: Theoretical values of $\tilde{H}(m)$ for the 512×512 realizations of the two Fourier methods when $H = 0.2$.

6. CONCLUSIONS

A new method called incremental Fourier synthesis was proposed to synthesize 2-D ESS process. The advantage of the method is that it is a relatively fast algorithm while it generates processes whose statistics virtually match those of true 2-D ESS processes. By choosing alternative forms of the structure function, an artist has precise control of the "roughness" of the texture with respect to scale. Furthermore, the algorithm can be extended to generate 3D (video) and even higher dimension ESS processes at the expense of $O(N^d \log_2(N))$ computations where d is the dimension.

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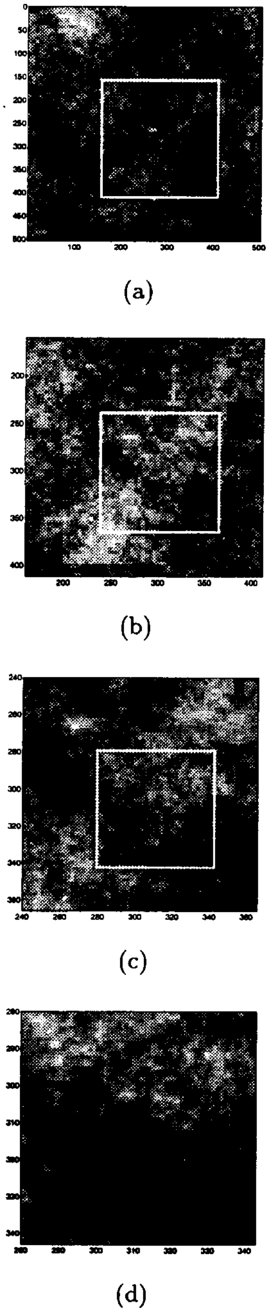

Figure 2: Zooming into a texture generated by standard Fourier synthesis with $H = 0.2$: (a) sampled every 8 units ($m = 3$), (b) sampled every 4 units ($m = 2$), (c) sampled every 2 units ($m = 1$) and (d) sampled every unit ($m = 0$).

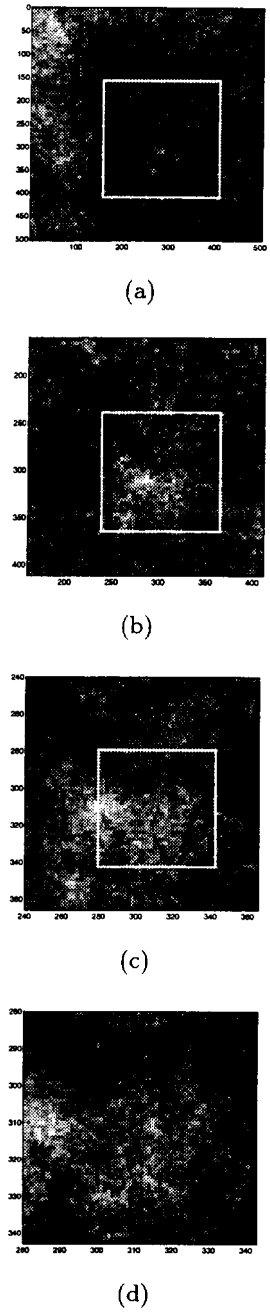

Figure 3: Zooming into a texture generated by incremental Fourier synthesis with $H = 0.2$: (a) sampled every 8 units ($m = 3$), (b) sampled every 4 units ($m = 2$), (c) sampled every 2 units ($m = 1$) and (d) sampled every unit ($m = 0$).