A PARAMETRICAL DESCRIPTION OF PLANE CURVES USING WAVELET DESCRIPTORS

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ABSTRACT

In this paper we propose and illustrate the application of Wavelet descriptors for the quantitative description of shapes. Fourier descriptors are known to be useful for describing shapes based on their boundaries. As these basically operate at one given scale or resolution, their application leads to loss of information of salient features of a shape contained at other scales.

Due to their inherent multiscale properties Wavelet descriptors are potentially suitable for shape discrimination in such situations. The paper gives the basic techniques of applying Wavelet descriptors in practical applications.

1. INTRODUCTION

Describing planar shapes by using a set of parameters is one of the most fundamental problems in pattern recognition. Several techniques have been suggested. We concentrate on the descrimination of shapes by using the boundary function γ.

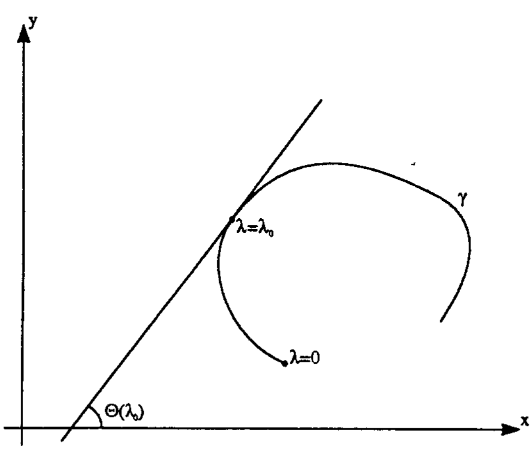

Figure 1: Parametric representation of a plane curve

The curvature function Φ is defined by

$$\Phi(\lambda) = \Theta(\lambda) - \Theta(0) , \qquad (1)$$

where λ is the arc length measured from the selected starting point (s. Fig 1).

One wants to obtain measures for characterizing shapes, i.e. measures which yield parameters close to one another in parameter space for similar shapes. Local features can be matched by using the Houghtransform [3]. This technique uses a-priori information as templates must be known. *Fourier descriptors* (FD's) as discussed in section 2 for this application have many desirable properties including invariance under translations, rotations and change in scale [2] [4]. But often one wants to discriminate shapes, that are very similar but differ in some detail. FD's fail for two reasons. First details contribute only to high frequencies which are very unspecific. Second FD's are *global* features whereas details are *local* properties of a shape. To overcome these disadvantages, we propose *Wavelet descriptors.*

Figure 2: Two similar shapes to be distinguished

2. FOURIER DESCRIPTORS

We follow the definition of Zahn and Roskies in [4]. Given the function Φ according to (1) we define

$$\Phi^*(\lambda) = \Phi\left(\frac{\lambda L}{N}\right) + \lambda \frac{2\pi}{N} , \qquad (2)$$

where L is the length of the boundary curve.

Expanding the periodic function Φ^* as a Fourier series

$$\Phi^*(\lambda) = \sum_{k=1}^{\infty} A_k \cos\left(\frac{k\lambda}{N} - \alpha_k\right) , \tag{3}$$

we obtain the FD's $\{A_k, \alpha_k\}$ for the curve γ. As shown in [4] Fourier descriptors posses a number of properties which correspond to the geometry of the shape, e.g. symmetric shapes have vanishing FD's at certain frequencies. As mentioned earlier, FD's fail in discriminating similar shapes as in Fig. 2.

3. WAVELET DESCRIPTORS

In this section we give the definition and an algorithm to compute Wavelet descriptors. Some properties of Wavelet descriptors are mentioned.

3.1. DEFINITION

The Wavelet descriptors $W^{\psi}_{a,b}$ of a given boundary function γ are defined as follows

$$W^{\psi}_{a,b}\{\gamma\} = \int_{-\infty}^{\infty} \Phi^*(\lambda)\psi_{a,b}(\lambda)\, d\lambda \tag{4}$$

with

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} \cdot \psi\left(\frac{x-b}{a}\right) \tag{5}$$

$$\Phi^*(\lambda) = \Phi\left(\frac{\lambda \cdot L}{N}\right) + 2\pi \cdot \frac{\lambda}{N} \tag{6}$$

$$N \geq L . \tag{7}$$

This means that the energy of ψ is constant for all scales and the length of γ is normalized to N. As Φ^* is periodic with N the Wavelet descriptors are periodic with respect to the shift b. Therefore we only consider the Wavelet descriptors with

$$0 \leq b \leq N . \tag{8}$$

3.2. CALCULATING WAVELET DESCRIPTORS

Considering that Wavelets are assumed to have compact support, there exists a s such that

$$\int_{|x|>s} |\psi(x)|^2 \; dx \to 0 \tag{9}$$

holds, i.e. s is the spatial width of ψ. Consequently we need considering only values of a with

$$0 < a \cdot s \leq N . \tag{10}$$

The limits of the integral in (4) can be changed to

$$W^{\psi}_{a,b} = \int_{-kN}^{kN} \psi_{a,b}(\lambda) \cdot \Phi^*(\lambda)\, d\lambda , \tag{11}$$

with k being an interger value

$$k > s + 1 . \tag{12}$$

Following eqn. (11) we calculate the Wavelet descriptors by partial integration. We obtain

$$W^{\psi}_{a,b}\{\gamma\} = [\Psi_{ab}(\lambda) \cdot \Phi^*(\lambda)]^{kN}_{-kN} - \int_{-kN}^{kN} \Psi_{a,b}(\lambda) \cdot \Phi'^*(\lambda)\, d\lambda \tag{13}$$

$$\Psi(\lambda) = \int_{-\infty}^{\lambda} \psi(x)\, dx \tag{14}$$

$$\Psi_{a,b}(\lambda) = \int_{-\infty}^{\lambda} \psi_{a,b}(x)\, dx = \sqrt{a} \cdot \Psi\left(\frac{\lambda - b}{a}\right) \tag{15}$$

$$\Phi^*(n \cdot N) = 0 \tag{16}$$

$$\Phi'^*(\lambda) = \frac{L}{N} \cdot \Phi'\left(\frac{\lambda L}{N}\right) + \frac{2\pi}{N} . \tag{17}$$

For the case that γ is a polygon

$$\Phi'(x) = \sum_{p=1}^{K} \Delta\varphi_p \cdot \delta(x - x_p) \tag{18}$$

holds (with x_p being the edges of one period of the polygon γ and $\Delta\varphi_p$ the angle at x_p) and we obtain the following formula for the Wavelet descriptors of γ

$$W^{\psi}_{a,b}\{\gamma\} = -\sum_{m=-k}^{k} \sum_{p=1}^{K} \Delta\varphi_p \cdot \Psi_{a,b}(\lambda_p - mN) - \frac{2\pi}{N} \cdot S(a) , \tag{19}$$

where

$$S(a) = \int_{-\infty}^{\infty} \Psi_{a,b}(x)\, dx \tag{20}$$

is a constant for each scale and

$$\lambda_p = x_p \cdot \frac{N}{L} . \tag{21}$$

3.3. HAAR WAVELET DESCRIPTORS

As an illustration we show the calculation of Wavelet descriptors based on the Haar Wavelet. We set the Haar Wavelet as

$$\psi(x) = h(x) = \begin{cases} 1 & \text{for} \quad -\frac{1}{2} \leq x \leq 0 \\ -1 & \text{for} \quad 0 \leq x \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad . \tag{22}$$

Thus

$$H(x) = \begin{cases} 1/2 + x & \text{for} \quad -\frac{1}{2} \leq x \leq 0 \\ 1/2 - x & \text{for} \quad 0 \leq x \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

$$S(a) = \frac{a \cdot \sqrt{a}}{4} \quad . \tag{24}$$

We obtain

$$W^h_{a,b} = -\sqrt{a} \sum_{m=-2}^{2} \sum_{p=1}^{k} \Delta\varphi_p \cdot H\left(\frac{\lambda_p - mN - b}{a}\right) - \frac{\pi}{2N} \cdot a\sqrt{a} \quad . \tag{25}$$

4. SOME PROPERTIES OF WAVELET DESCRIPTORS

Wavelet descriptors yield some properties reflecting the shape's geometric features.

Symmetric shapes

If the shape is symmetric with respect to a point t

$$\Phi^*(t - \lambda) + \Phi^*(t + \lambda) = \Phi^*(t^+) + \Phi^*(t^-) \tag{26}$$

and the Wavelet function is symmetric

$$\psi(x) = \psi(-x) \tag{27}$$

then

$$\forall a : \quad W^\psi_{a,t}\{\gamma\} = 0 \quad . \tag{28}$$

If the shape is symmetric under rotation

$$\Phi^*(\lambda) = \Phi^*(\lambda - t) \tag{29}$$

and the wavelet yields

$$\psi(x) = 0 \quad \text{for } |x| > c \tag{30}$$

$$\psi(x) = -\psi(x - c) \quad \text{for } x \in [0; c] \tag{31}$$

(e.g. the Haar Wavelet holds these conditions) then

$$W^\psi_{a,b} = 0 \quad \text{for } a = \frac{t}{c} \quad . \tag{32}$$

Integration in one scale

The integral over one period N at one scale with respect to the shift b is zero, i.e.

$$\int_0^N W_{a,b}(x)\, db = 0 \quad . \tag{33}$$

This means that the function

$$\varphi_a(x) = \begin{cases} W^\psi_{a,x} & \text{for} \quad x \in [0; N] \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

is a wavelet. This wavelet contains information about the shape γ and can therefore be used for matching other shapes.

5. DYADIC WAVELET DESCRIPTORS

The continous Wavelet transform as used above is not suitable as the coefficients contain redundant information. Therefore we use dyadic Wavelets for the calculation of *dyadic Wavelet desriptors.*

The dyadic Wavelet descriptors of a shape are defined by

$$W^\psi_{m,n} = \int_{-\infty}^{\infty} \Phi^*(\lambda) \cdot \psi\left(\frac{\lambda - n}{2^m}\right) d\lambda \tag{35}$$

$$m \in \mathcal{N} \tag{36}$$

$$n = 0 \cdot 2^m \, , \, 1 \cdot 2^m \, , \, 2 \cdot 2^m \, , \, \ldots \tag{37}$$

The length N of Φ^* has therefore to be normalized to a power of 2.

6. USING WAVELET DESCRIPTORS FOR SHAPE DESCRIMINATION

We calculate the Fourier descriptors and the dyadic Wavelet descriptors of the two shapes in Fig (2). Length of both shapes was normalized to $N = 128$. The Fourier descriptors are shown in Table (1) and the Wavelet descriptors in Table (2).

The Fourier descriptors shown differences in all frequencies. There are some ($n = 4, 8, 12$) that have similar values which corresponds to the square-like shapes. But the differences cannot be localized.

The Haar Wavelet descriptors show that the shapes are both symmetric at large scales (i.e. the 'notch' in shape 1 is not regarded). But differences appear in other scales. The Wavelet descriptors can be analysed in a better way, if their values are normalized in every scale.

	shape 1		shape 2	
n	$\|A_n\|$	φ_n	$\|A_n\|$	φ_n
1	0.0213	-1.8266	0.0167	-0.7165
2	0.0739	-0.1615	0.0004	1.8116
3	0.0141	1.1490	0.0239	0.4065
4	0.4766	-1.9416	0.4887	-1.9469
5	0.0343	3.0740	0.0088	-0.9259
6	0.0860	-0.8429	0.0016	1.9725
7	0.0478	0.4333	0.0300	0.0680
8	0.2216	-2.4747	0.2279	-2.3250
9	0.0408	-2.6636	0.0027	-0.0377
10	0.0902	-0.6942	0.0039	1.8475
11	0.0328	1.3796	0.0347	-0.2840
12	0.1602	-2.6103	0.1350	-2.7074
13	0.0303	0.7655	0.0085	0.8755
14	0.0249	-1.6878	0.0069	1.6139
15	0.0365	-1.7761	0.0378	-0.6451

Table 1: Fourier descriptors of the shapes in Fig (2)

7. CONCLUSION

Wavelet descriptors have been introduced based on the Haar Wavelet. Future research will concentrate on using various mother wavelets with an increased number of vanishing moments as then their coefficients contain more information [1].

Another approach that leads to parametrisation of the boundary function γ is using R-tables. This works especially good for convex hulls. By taking the Wavelet transform of this function $r(\varphi)$ Wavelet descriptors are obtained that have a more specific meaning in the spacial domain.

The main advantage in comparison with Fourier descriptors is the constant bandwidth at each resolution. Fourier descriptors will perform better for regular shapes when details can be neglegted. Wavelet descriptors allow a zoom-in at details using the same technique as for describing global features.

A better understanding of what *'bandwidth'* or *'frequency'* means in terms of shapes will be necessary to interpretate the coefficients of both Fourier and Wavelet descriptors.

a	b	$W_{a,b}(shape1)$	$W_{a,b}(shape2)$
64	0	0.1077	1.470
	64	-0.1077	-1.470
32	0	0.4009	0.1039
	32	-0.4388	0.0148
	64	0.2486	-0.1039
	96	-0.2107	-0.0148
16	0	1.8662	1.7739
	16	-1.2992	-1.6269
	32	1.2992	1.6269
	48	-1.9198	-1.6059
	64	1.9198	1.6059
	80	-1.5682	-1.7529
	96	1.5682	1.7529
	112	-1.8662	-1.7739

Table 2: Wavelet descriptors of the shapes in Fig (2)

8. REFERENCES

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