TEXTURE REPRESENTATION THROUGH MULTIFRACTAL ANALYSIS OF OPTICAL MASS DISTRIBUTIONS

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ABSTRACT

The paper addresses the analysis of singular distributions defined on a fractal support, called fractal measures. In general, a fractal measure has an infinite number of singularities of infinitely many types. The term *multifractals* expresses the fact that points, corresponding to a given type of singularity, tipically form a fractal subset whose dimensions depend on the type of singularity. The theory of the q-th order generalized fractal dimensions supplies a tool for the characterization of such multifractal measures. This theory results from an extension of the fractal dimension to different-order statistics. The paper exploits such concepts in order to face the problem of texture recognition. In particular the fractal measure taken into account is the 2D distribution of the optical mass of an image; some theoretical aspects related to this problem are addressed. Results on real images are presented and discussed.

1. THE BINARY MULTIPLICATIVE PROCESS

Multifractality is a property relevant to objects with subsets whose fractal dimensions do not equal the global one. So, to characterize the multifractality of a set, one has to find out the spectrum of the fractal dimensions of the subsets different from the one of the global set; in short, the multifractality spectrum. Let suppose to have a segment S of size $s(S)=1$, over which a set of P points is distributed. Let divide the segment into $N=2^n$ equal parts; let call N_i the number of points contained in the i-th partition, and μ_i the percentage contained in it. If we consider P and n increasing to infinite, the set $M=\{\mu_i\}$ ($i=0,...,N$) contains all the information about the distribution of the points over S. Let now consider a binary multiplicative process that generates a distribution of points. These P points are distributed as follows: at first, let divide S into two equal parts and distribute over each a different percentage of the initial population (that is, μ_0, μ_1, where $\mu_0+\mu_1=1$); after, let repeat recursively the above procedure for each subpart of the segment; in other words, let re-distribute the population, present over that sub-segment, over the two halves. At the n-th iteration a set M characteristic of the distribution is obtained, $M=\{(\mu_0)^n,(\mu_0)^{n-1}\mu_i,...,(\mu_1)^n\}$. By counting the intervals, over which a portion of the population equal to $\rho_k=(\mu_0)^{n-k}$ is distributed, they result to be $N(k)=n!/[k!(n-k)!]$, with $k\in(0,...,n)$. As we have to calculate the limit for (n,P) increasing to infinite, it is useful to define $\xi=k/N$ and solve in function of such a variable. Hence, we obtain $N(\xi)=n!/[(n\xi)!(n-n\xi)!]$ and $\rho_k=\{(\mu_0)^{1-\xi}(\mu_1)^{\xi}\}^n$. Let consider now the set of intervals (sets) with population ρ_ξ, called S_ξ, and look for its fractal dimension

$$M_d(S_\xi)=\lim_{\delta\to 0}\sum_{S_\xi}\delta^d=\lim_{\delta\to 0}N(\xi)\delta^D=\begin{cases}0 & d>D(\xi)\\ +\infty & d<D(\xi)\end{cases}$$

By computing such limit with $\delta=2^{-n}$ (with $n\to+\infty$), and by using the Stirling relation $n!\cong(2\pi\cdot n^{n+1})^{0.5}\cdot e^{-n}$ one obtain

$$N_n(\xi)\approx\left[2\pi n\xi(1-\xi)\right]^{-0.5}e^{-n[\xi\log\xi+(1-\xi)\log(1-\xi)]}$$

$$D(\xi)=f(\xi)=-(\ln 2)^{-1}\xi\ln\xi+(1-\xi)\ln(1-\xi)$$

It is possible to observe how the sets S_ξ characterize the distribution of points over S, as all of them are sets of points with same densities and $S=\mathbf{Y}_\xi\ S_\xi$. Moreover, even if the fractal dimension of S is equal to 1 (it is a segment), the fractal dimensions $D(S_\xi)$ assume values between 0 and 1; each subset has its own fractal dimension, and from this has origin the concept of multifractality. Instead of the parameter ξ, it is better to use the Lipschitz-Holder exponent α, which definition comes from $\rho_\xi=\delta^\alpha$. By considering $\delta=2^{-n}$, one obtain

$$\alpha=\ln\rho_\xi/\ln\delta=(\ln 2)^{-1}\xi\ln\rho_\xi+(1-\xi)\ln(1-\rho_\xi)$$

It has to be observed that α is defined in $[\alpha_m, \alpha_M]$, as $\xi\in[0,...,1]$, where $\alpha_m=-\ln(1-\rho_\xi)/\ln 2$ and $\alpha_M=-\ln\rho_\xi/\ln 2$; $f(\alpha)=f[\alpha(\xi)]$ is the fractal dimension associated to the set S_α.

2. THE $\tau(q)$ AND $D(q)$ FUNCTIONS

When analyzing discrete sets, the simplest way to manage them is offered by the so-called *box-counting* method, which consists in dividing the sets into boxes. However, in such a way, one loses many information contained in the set, as the distribution of points; so, it is necessary to consider not only the boxes covering the set, but also the masses that they contain. Hence, to each box it is associated a quantity $\mu_i=N_i/N$ where N is the total number of points in the set, and N_i is the number of points contained into the i-th box. Hence, one has the set $M=\{\mu_i\}$, where $i=0,...,B$; B is the number of boxes that are needed to cover the set. D_b is the box-dimension of S, $D_b=\lim_{\delta\to 0}\ln B/\ln\delta$. In order to weight differently the subset of S, depending on the masses contained in the boxes, one has to measure S and to compute $M_d(q,\delta)=\sum_{i=0,...,N(\delta)}\mu_i^q\delta^d=N(q,\delta)\delta^d$ where

$$\lim_{\delta\to 0}M_d(q,\delta)=\begin{cases}0 & d>\tau(q)\\ +\infty & d<\tau(q)\end{cases}.$$

In the box-dimension computation, $\tau(q)=-\lim_{\delta\to 0}\{\ln[N(q,\delta)]/\ln\delta\}$

In this way, depending on the values of q, the different subsets have different weights; this fact can also be evidenced by the behavior of $\tau(q)$ at the boundaries of the existence range. If the maximum and minimum values assumed by the densities are indicated by μ_m and μ_M, and N_m and N_M the number of boxes in the same cases, the infinite limit

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$$\lim_{q\to-\infty} \tau(q) \approx -\lim_{\delta\to 0}\lim_{q\to-\infty} \left(\ln \sum_{i=0,\dots,N_m} \mu_i^q / \ln\delta\right) = \lim_{\delta\to 0}\lim_{q\to-\infty} q(\ln N_m/\ln\delta)$$

behaves as a straight line. By considering the derivative of $\tau(q)$

$$\mathrm{d}\tau(q)/\mathrm{d}q = -\lim_{\delta\to 0}\left(\sum_i \mu_i^q \ln\mu_i/\ln\delta - \sum_i \mu_i^q\right)$$

$$\lim_{q\to-\infty} \left(\mathrm{d}\tau(q)/\mathrm{d}q\right) = -\lim_{\delta\to 0}\left(\ln\mu_m/\ln\delta\right) = -\alpha_M$$

$$\lim_{q\to+\infty} \left(\mathrm{d}\tau(q)/\mathrm{d}q\right) = -\lim_{\delta\to 0}\left(\ln\mu_M/\ln\delta\right) = -\alpha_m$$

Moreover, $\tau(0)$ is the fractal dimension of the set under examination, and $\tau(1)=0$ as the $\{\mu_i\}$ are normalized to the unit. In the case of the quaternary multiplicative process used before, it is possible to find out the analitical form of $\tau(q)$ when $\delta=2^n$

$$N(d,\delta) = \sum_{k=0}^{n}\left[n!/k!(n-k)!\right] p^{qk}(1-p)^{q(n-k)} = \left[p^q + (1-p)^q\right]^n$$

from which $\tau(q) = \ln\left[p^q + (1-p)^q\right]/\ln 2$. The function $\tau(q)$ is linked to $f(\alpha)$; from its definition, to α correspond the boxes μ_α so that $\mu_\alpha = \delta^{-\alpha}$ and that the set of these boxes has, for $\delta\to 0$, a fractal dimension equal to $f(\alpha)$. By calling $N(\alpha,\delta)$ the number of boxes needed to cover the set S_α, with $\alpha\in(\alpha,\alpha+\mathrm{d}\alpha)$, this number result to be, if δ is sufficiently small, $N(\alpha,\delta)=\rho(\alpha)\delta\text{-}f(\alpha)\mathrm{d}\alpha$, where $\rho(\alpha)$ is the number of sets from S_α to $S_{\alpha+\mathrm{d}\alpha}$. From its definition, $M_d(q,\delta)$ results to be equal to the summation of $(\mu_i)^q$, and consequently, in the continuous case,

$$M_d(q,\delta) = \int \rho(\alpha)\delta^{-f(\alpha)}\delta^{\alpha q}\mathrm{d}\alpha = \int \rho(\alpha)\delta^{-\left[f(\alpha)-\alpha q-d\right]}\mathrm{d}\alpha$$

In the limit for $\delta\to 0$, the integral behaves as the maximum on α of the argument; of this one, the factor $\rho(\alpha)$ remains bounded, as not depending from δ, and then the dominant term is $\alpha(q)$; so that, when $\alpha=\alpha(q)$, $\mathrm{d}[q\alpha\text{-}f(\alpha)]/\mathrm{d}\alpha=0$ and, for this value,

$$M_d(q,\delta) = K\delta^{-\left[f(\alpha)-\alpha q-d\right]}$$

Hence, there is a relation between $\tau(q)$ and $f(\alpha)$, which can be expressed in a parametric form as $\tau(q)=f[\alpha(q)]-q\alpha(q)$. By deriving, we obtain that, when $\alpha=\alpha(q)$,

$$\frac{\mathrm{d}\tau(q)}{\mathrm{d}q} = \frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}q} - q\frac{\mathrm{d}\alpha}{\mathrm{d}q} - \alpha = \frac{\mathrm{d}\alpha}{\mathrm{d}q}\left[\frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha} - q\right] - \alpha = -\alpha$$

If one wants a function constant over an E-dimensional space, and equal to E, one has to use the function $D(q)$, defined as $D(q)=\tau(q)/(1\text{-}q)$. With this kind of definition, in order to obtain a function continuous and defined over $q\in\Re$, it is necessary to manage differently the point $q=1$

$$\lim_{q\to 1} D(q) = -\lim_{\delta\to 0}\lim_{q\to 1}\frac{1}{1-q}\frac{\ln\sum_i \mu_i^q}{\ln\delta} = -\lim_{\delta\to 0}\frac{\sum_i \mu_i \ln\mu_i}{\ln\delta}$$

by using the Hospital theorem. Hence, it is possible to make the function continuous by defining $D(q)$ as follows

$$D(q) = \frac{\tau(q)}{1-q} \text{ if } q\neq 1; \qquad D(q) = -\lim_{\delta\to 0}\frac{\sum_i \mu_i \ln\mu_i}{\ln\delta} \text{ if } q=1$$

The value q=1 is proportional to the entropy H of the distribution. In particular, $H\equiv -D_1\log\delta$. By considering an E-dimensional space over which it is distributed a population of points in an uniform way, and by dividing the space into $N=\delta^{-E}$ cells, each one containing a portion $\mu_i=\delta^{-E}$ of points. It results $N(q,\delta)=\delta^{(q-1)E}$ and then

$$D(q) = -(1-q)^{-1}\lim_{\delta\to 0}\left\{\ln\delta^{(q-1)E}/\ln\delta\right\} = E$$

From the definition, it comes that $D(q):\Re\to\Re$ and that, by using the same approach as for $\tau(q)$

$$\lim_{q\to-\infty} D(q) = -\lim_{\delta\to 0}\lim_{q\to-\infty}(1-q)^{-1}\ln\sum_i \mu_i^q/\ln\delta = \alpha_M$$

Analogously, $\lim_{q\to+\infty} D(q) = \alpha_m$. Moreover, $D(0)$ is the fractal dimension of the set, if we assume, by definition, the value equal to $\tau(0)$. Hence, the function $D(q)$ is not increasing, has horizontal asymptotes to the infinite, is always positive, and, in q=0, equals the fractal dimension of the set. If we consider once again the case of the binomial multiplicative process, it is possible to compute this function in an analytical way, resulting

$$D(q) = \ln\left[p^q + (1-p)^q\right]/(1-q)\ln 2$$

It is constant if and only if p=0.5; otherwise, it is like in Fig.1.

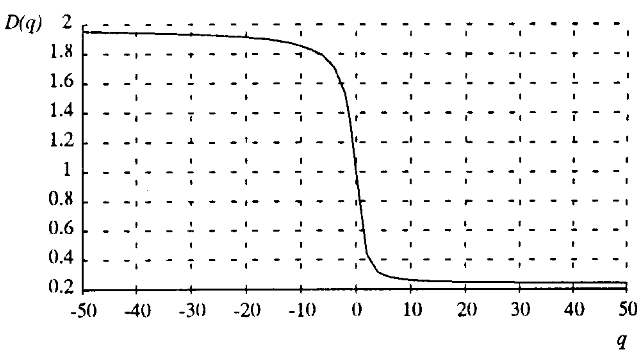

Fig.1. $D(q)$ behavior for the binomial process.

3. TEXTURE ANALYSIS AND MULTIFRACTALS

A class of measure-based techniques for fractal-dimension estimation of real textures is based on the box-dimension concept. Voss demonstrated [2] that $N(\delta)\propto\sum_m m^{-1}\mathrm{P}(m|\delta)$, where $\mathrm{P}(m|\delta)$ is the probability to find m points within a box of side δ centered on a generic point of the set under examination. From this

$$D = -\lim_{\delta\to 0}\left\{(\ln\delta)^{-1}\ln\left[\sum_m m^{-1}\mathrm{P}(m|\delta)\right]\right\}$$

The box-dimension so computed is not dependent on the grid, as in the box-counting method, and the resulting $N(\delta)$ is a real number. However, although very simple to use, the box-counting method has many drawbacks that make it not so attractive for discrete signal processing. In particular, the results are strongly dependent on the origin of the partitioning grid. Hence, a different definition of $D(q)$ is used that extends the method introduced by Voss to estimate the fractal dimension. By following the Voss' approach, it is possible to extend the concept of d-measure $M_d(\delta)$, to q-th order momenta, as

$$M_d(q,\delta) = \sum_m m^q \mathrm{P}(m|\delta)\delta^d = N(q,\delta)\delta^d$$

and to compute the q-th order fractal dimensions

$$D'(q) = q^{-1} \lim_{\delta\to 0} \left\{ \ln\left[\sum m^q \mathrm{P}(m|\delta)\right] / \ln\delta \right\} \quad \text{if} \quad q \neq 0$$

$$D'(q) = \lim_{\delta\to 0} \left\{ \sum_m \left[(\log m)\mathrm{P}(m|\delta)\right] / \log\delta \right\} \quad \text{if} \quad q = 0$$

It can be demonstrated that, in the case of point distributions where each point has a unique mass $1/N_t$, where N_t is the number of points of the whole set, the latter definition of $D(q)$, named $D'(q)$, is equivalent to the former, as $D(q+1)=D'(q)$. These concepts are based on the hypothesis that the analyzed sets are continuous or dense; so, it is possible to reach measuring-scale dimensions that are infinitesimal. Moreover, these sets fit the multifractal model at all scales; it is possible to define a multifractal measure [1]. In the case of real structures, it is mandatory to define a different and approximated multifractal model to estimate the function $D(q)$. In other words, a model must be defined for a given range of scales, and a tool for parameter estimation must be provided. Let us consider a set S that complies with the multifractal model $D'(q)$ if, for a small δ value (i.e., $\delta \ll s(S)$, where $s(S)$ is the set size), there exist (δ_m, δ_M) such that $N(q,\delta) \cong k_q \delta^{qD(q)}$, where $\delta \in (\delta_m, \delta_M)$. Now it is possible to estimate $D'(q)$ through a linear regression in the $\{\ln\delta, \ln N(q,\delta)\}$ bilogarithmic plane, applied to samples in the linear region. This implies that, in order to estimate the fractal dimension, it is necessary to detect the range (δ_m, δ_M) over which the behavior is linear in the same bilogarithmic plane as defined above.

4. MULTIFRACTAL TEXTURE MEASURES

It is possible to define different multifractal measures related to a generic signal $f(x)$, $f:\Re^E \to \Re$. One can take into account the signal graph, that is a multidimensional $(E+1)$ surface, or consider the image as a optical mass bi-dimensional distribution. In the former, a major problem in the estimation arises from the fact that a digital image is a discrete set of points, and this gives rise to a drawback for negative q values; i.e., small masses predominate over large ones, and, for small δ values, the measures of small masses are less accurate. This drawback produces in the bi-logarimic plane two linear regions with two different slopes (Fig.2). The samples for small δ values are totally dominated by very few numbers (usually, $\mathrm{P}(\delta|1)$ is not null for small δ) and the fractal behavior is masked. To evaluate the linearity of the set of measures, the parameter $I=[4(\mu_{xy})^2+(\mu_{xx}-\mu_{yy})^2]^{0.5}/(\mu_{xx}+\mu_{yy})$ is defined, where μ_{ij} denotes the covariance of the points in the set. The values of the parameter I are included in the range [0,1], and the maximum value is reached when, and only when, all data lie on a straight line. The scale range (δ_m, δ_M) should be determined for each q value in order to optimize the linearity, or better, to identify the range in which the multifractal model is verified; to this aim, the parameter I is estimated over a moving window, with dimension (δ_i, δ_{i+k}), along such data distribution in the bilogarithmic plane, so producing an estimation of the local linearity $L(i)$

$$L(i) = I\left\{\left[\ln\delta_i, \ln N_q(\delta_i)\right], \ldots, \left[\ln\delta_{i+k}, \ln N_q(\delta_{i+k})\right]\right\}$$

where k is experimentally set to 3÷5. The behavior of $L(i)$ shows a plateau bounded by one or two minima, which represent the upper and lower bounds for the linear region, and are assigned to δ_m and δ_M (Fig.3).

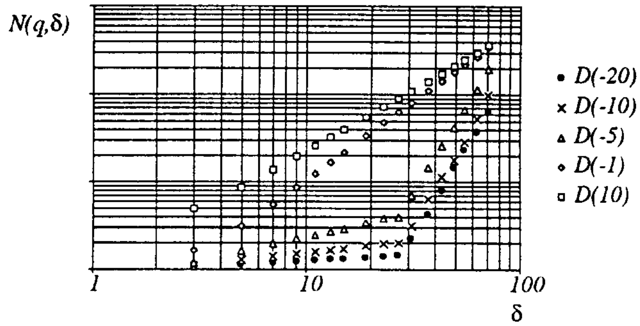

Fig.2. Samples in the bi-logarithmic plane.

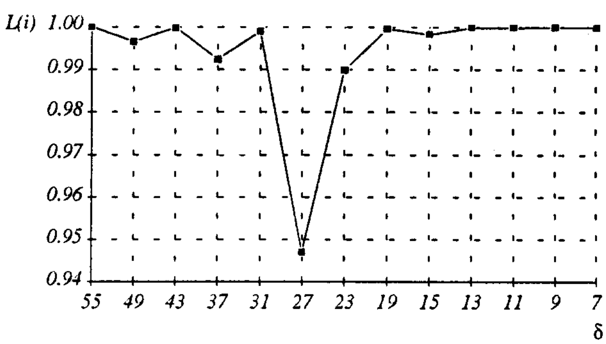

Fig.3. The behavior of $L(i)$.

However, in small textured areas there are few scales of measure only and, as for negative q values the number of scales is reduced, it is not possible to compute a reliable $D(q)$. Therefore, for negative q values in $D'(q)$, the resolution should be increased through interpolation in the case of small measuring-scales δ. This can be done by: (i) using a smoothing interpolant, which minimizes the mean square error and gives a stable measure; (ii) using an interpolant complying with the multifractal model, based, for instance, on the iterated function systems. Fractal interpolation showed the unlike property of forcing the set to be a single fractal, as its application results in flatting the $D'(q)$ behavior. The bilinear interpolant is also varying this behavior, but it results only in lowering the fractal dimension $D'(-1)$, without changing the asymptotes and the derivative of $D'(q)$ in the flexus. So, the bilinear interpolation has been preferred. To sum up, the estimation algorithm includes three steps: (i) the distribution $\mathrm{P}(m|\delta)$ is computed for boxes of size $\delta = Bs^i$, where $i \in \{0,1,2,\ldots\}$; s is a number between 0 and 1 (and experimentally choosen between 0.8 and 0.9); and B is the size (in pixels) of the largest box (in this way, a linear series in the bilogarithmic plane is obtained); (ii) for each q value, the scale range and the resolution are computed; (iii) an interpolation of multiple δ values is performed to estimate the limit of $\delta \to 0$ and the correct $D'(q)$ value. Images are considered as mass distributions generated by a subdivision process $F(p,r)$; $p=(p_1,...,p_n)$ is the vector of the probabilities of mass distributions, and $r=(r_1,...,r_n)$ is a space partitioning vector. Consequently, each pixel has a mass $f(x,y)$ equal to its gray level. Optical images can be considered samples of the continuous radiance function. Under these conditions, such a measure is not a multifractal one and leads to a trivial (uniform) $D(q)$ function. However, images have discontinuities in the first derivatives, thus, the $D(q)$ function is computed for the image of the gradient. For non-optical images with high discontinuities (SAR), the multifractal parameters can be estimated directly on original images. It is necessary to have a set where every point has a unique mass value in order to apply the latter definition of $D(q)$. Therefore, each pixel (x,y) with mass $f(x,y)$ is assumed to be the overlap of $n=f(x,y)$ points with unitary mass. If compared with the previous one, the $P(m|\delta_i)$ estimation algortithm is very fast: for optical images, the gradient is computed; afterwards, the image $m(x,y,\delta_i)$ is obtained by filtering the image $f(x,y)$ with a moving average. Finally, for each $m(x,y,\delta_i)$ the correspondig entry $P\{m(x,y,\delta_i)\}$ of the histogram is increased proportionally to $f(x,y)$, i.e., the original gray value of the pixel.

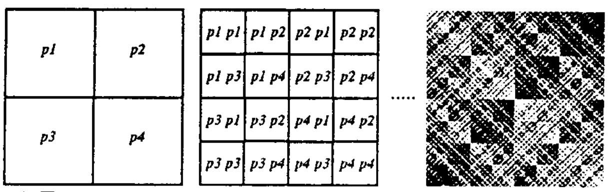

Fig.4. The recursive subdivision algorithm.

Through such definition, it is also possible to generate test images characterized by an analitically known $D(q)$. By using a recursive mass subdivision schema (Fig.4), one can obtain a fractal distribution denoted by (where $\Sigma_i\, p_i=1$)

$$D(q) = (q-1)^{-1} \ln \Sigma_i\, p_i^q / \ln 2$$

and estimation results very close to the analitical ones (Fig.5).

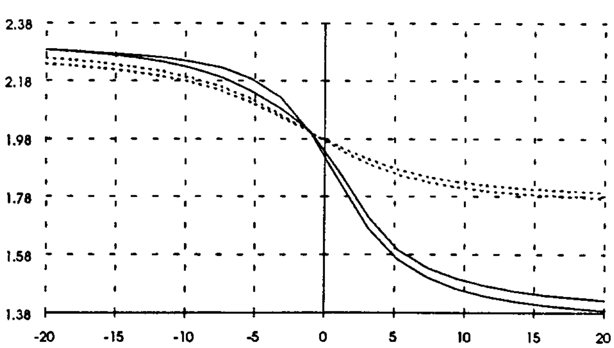

Fig.5. Analitical and estimated $D(q)$ functions.

5. RESULTS

The functions $D(q)$ were estimated on SAR textures by using the mass distribution algorithm. 128×128 windows were analyzed, using 15 different measuring scales (i=0,1,...,14, ranging from 3 to 61 pels, with s=0.87). The $D(q)$ behaviors are shown in terms of mean value and standard deviation. Even though some overlaps occur for certain ranges, significant differences are found for the other ranges, so allowing an easy texture discrimination.

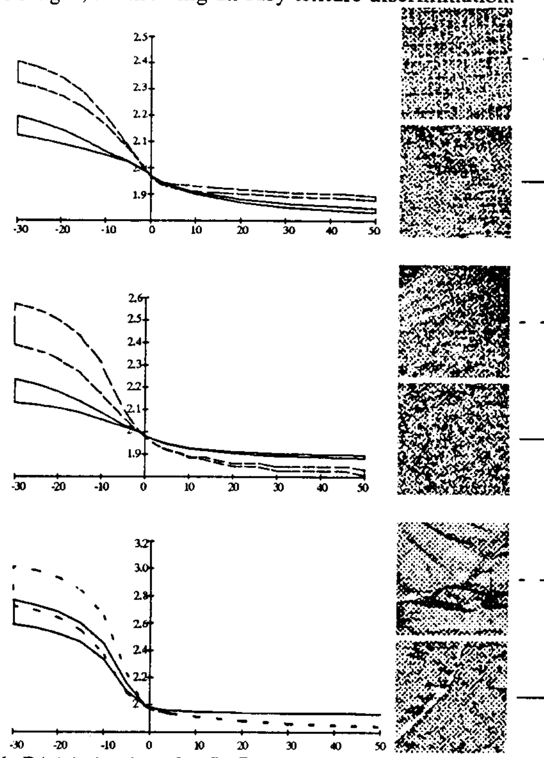
Fig.6. $D(q)$ behaviors for SAR textures.

Brodatz's texture classification was also performed; four multifractal features were used: $D(-3)$ and $D(3)$; the first derivative in q=0; the fractal dimension of the 3D surface. A k-NN classifier was used and the relevant results were compared with classical co-occurrence matrix analysis. Each 256×256 image was partitioned in overlapped 64×64 windows (one every 32 pixel in both x and y directions); results are reported in Fig.7.

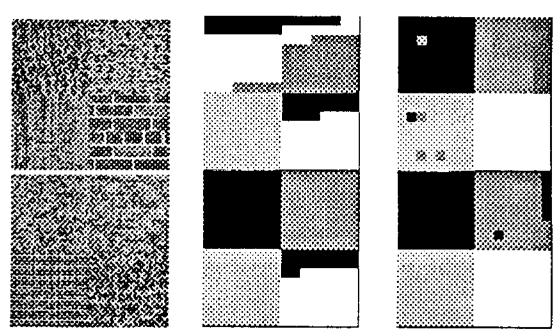
Fig.7. Some Brodatz textures (left); classification results by the cooccurence matrix approach (middle): error 15%; classification results by the multifractal approach (right): error 2%.

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