AN UPPER BOUND ON THE AREA OCCUPIED BY A FRACTAL

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ABSTRACT

Fractal images defined by an iterated function system (IFS) are specified by a finite number of contractive affine transformations. In order to plot the image specified by the transformations on the screen of a digital computer, it is necessary to determine a bounding area for the image. This paper derives a formula that expresses the dimensions of this bounding area in terms of the transformations.

1. INTRODUCTION

Iterated function system (IFS) fractal images, as popularised by Barnsley[1, 2], are constructed from sheared, reduced, rotated and displaced copies of themselves. For example, the *curly* image in Figure 1 is constructed from two transformed copies of itself: the blackened rightmost curl and the less black remainder. The blackened rightmost curl is a reduced, rotated and displaced copy of the whole image, produced by applying transformation T_1. The remainder of the image is produced by applying transformation T_2, which shrinks the whole image and rotates it anti-clockwise.

Various algorithms exist for plotting fractal images from their affine transformations[3, 4, 5]. However, in order to plot a fractal image on the screen of a digital computer, all of these algorithms require advance knowledge of a *bounding area* inside which the image is known to lie. In practice, this bounding area is usually determined by trial and error, as none of the main references[2, 6] give a formula for determining a bounding area for a fractal from its transformations.

In this paper we derive a formula which expresses the dimensions of a bounding circle for a fractal image in terms of the transformations defining the fractal. In Section 3 we derive an upper bound for the special case of a fractal defined by two transformations by considering how the transformations interact. In Section 4 we derive a more general formula by considering the effect of each transformation individually on the centre of the bounding circle, as proposed in [7]. We find that the second derivation gives a tighter bound provided the centre of the bounding circle is located in an appropriate position.

2. DEFINITIONS OF FIXPOINT AND SCALING FACTOR

The derivations use two properties of an affine transformation: its *fixpoint* and its *scaling factor*. An affine transformation defined by the equation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$ has a fixpoint $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ that is mapped to itself under the transformation, i.e. $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$. Because the transformations are contractive, repeated application of a transformation to any point eventually leads to the fixpoint of the transformation. For example, for *curly*, the fixpoint of transformation T_1 is the point where the blackened rightmost curl of the blackened rightmost curl (of the blackened rightmost curl of ...) shrinks to a single point and the fixpoint of transformation T_2 is in the centre where repeated application of T_2 shrinks the whole image to a single point. Solving the equation defining $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ gives $x_0 = \dfrac{bf + e(1-d)}{(1-a)(1-d) - bc}$, $y_0 = \dfrac{ce + f(1-a)}{(1-a)(1-d) - bc}$.

Note that we can translate the origin to the fixpoint of a transformation: the coefficients of the transformation about the new origin become a', b', c', d', e', f', where $a' = a, b' = b, c' = c, d' = d, e' = 0, f' = 0$. In this case applying the transformation to a point moves it closer to the origin.

The scaling factor of a transformation T_i is given by $s_i = \sqrt{\alpha + \beta + \sqrt{(\alpha - \beta)^2 + \gamma^2}}$ where $\alpha = (a^2 + c^2)/2, \beta = (b^2 + d^2)/2, \gamma = ab + cd$[8].

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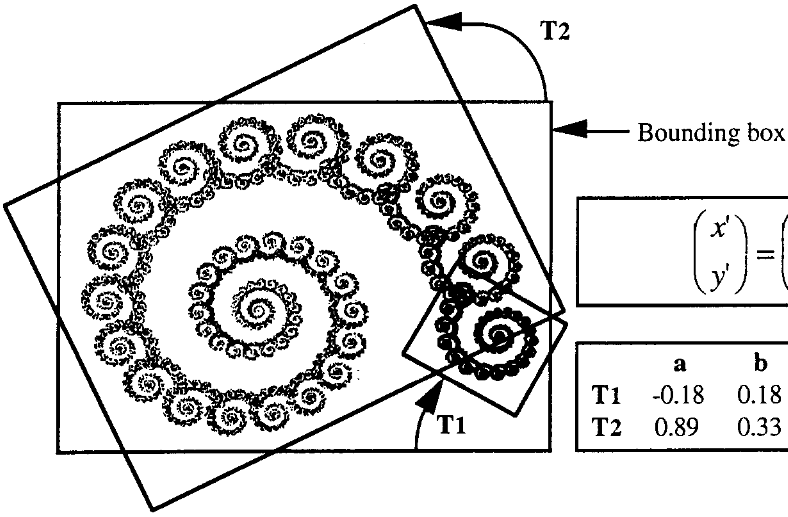

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

	a	b	c	d	e	f
T1	-0.18	0.18	-0.18	-0.18	0.83	0.10
T2	0.89	0.33	-0.33	0.89	0.0	0.0

Figure 1 The fractal image *curly* and its transformations.

The significance of s_i is that two points with separation R are mapped by T_i to points whose separation is at most $s_i R$. In particular, given a circle C_1 with radius R_{C_1} centred on the fixpoint of T_i, we can construct a circle C_2 with radius $s_i R_{C_1}$ also centred on the fixpoint of T_i, such that T_i maps every point inside C_1 to a point inside C_2.

3. A BOUNDING CIRCLE FOR A FRACTAL DEFINED BY TWO TRANSFORMATIONS

Suppose we are given a fractal defined by two transformations T_1 and T_2. Suppose T_1 has the fixpoint F_1 and the scaling factor s_1 and T_2 has the fixpoint F_2 and the scaling factor s_2. We denote the distance between F_1 and F_2 by F_1F_2. We assume $s_1, s_2 < 1$ and $s_1 \geq s_2$.

We wish to construct a circle B_1 with radius R_{B_1} that is a bounding circle for the fractal image. We can assert that B_1 is a bounding circle if both T_1 and T_2 map every point inside B_1 to another point inside B_1. We construct B_1 in the following way to satisfy this constraint.

- B_1 is centred on F_1: as T_1 is contractive, it maps every point inside B_1 to another point inside B_1.
- We construct a circle B_2 containing B_1, such that T_2 maps every point inside B_2 to a point inside a circle B_3 contained by B_1. Thus T_2 also maps every point inside B_1 to another point inside B_1.

The construction proceeds as follows. Figure 2 illustrates the result.

1. We consider a circle B_1 centred on F_1 with radius R_{B_1}. As $s_1 < 1$, T_1 maps every point inside B_1 to another point inside B_1.
2. The extension of the line F_2F_1 intersects B_1 at the point P_1. P_1 is therefore the point on B_1 furthest from F_2: the distance F_2P_1 is $R_{B_1} + F_1F_2$. Thus a circle B_2 centred on F_2 with radius $R_{B_1} + F_1F_2$ contains B_1.
3. The extension of the line F_1F_2 intersects B_1 at the point P_2. P_2 is therefore the point on B_1 closest to F_2: the distance F_2P_2 is $R_{B_1} - F_1F_2$. Thus a circle B_3 centred on F_2 with radius $R_{B_1} - F_1F_2$ is contained by B_1.
4. The ratio of the radii of B_3 and B_2 is $\dfrac{R_{B_1} - F_1F_2}{R_{B_1} + F_1F_2}$ and they are both centred on F_2. By suitable choice of R_{B_1} we can assert $\dfrac{R_{B_1} - F_1F_2}{R_{B_1} + F_1F_2} = s_2$, when T_2 maps every point inside B_2 to a point inside B_3. As B_2 contains B_1 and B_1 contains B_3, T_2 maps every point inside B_1 to another point inside B_1.

The circle B_1 centred on F_1 is therefore a bounding circle for the fractal. Solving for its radius R_{B_1} gives

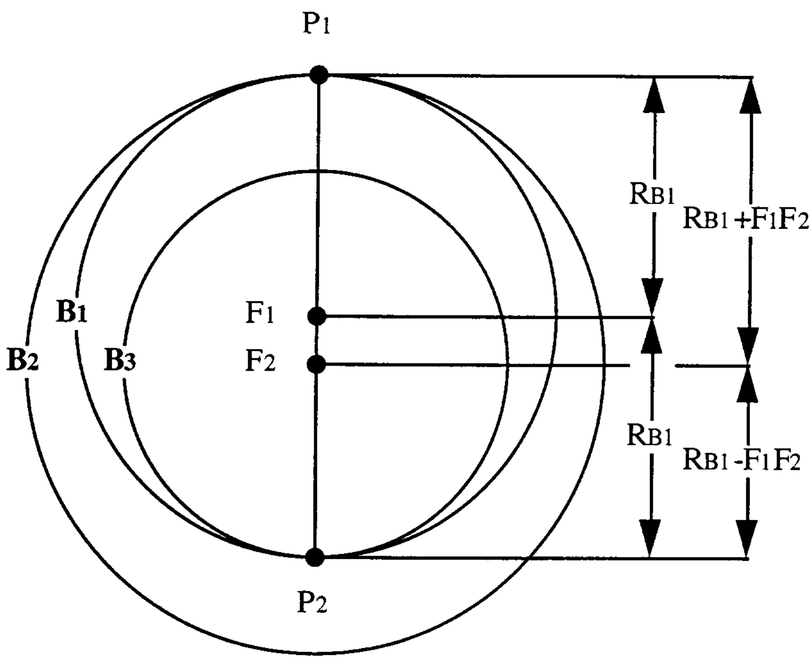

Figure 2 The circles B_1, B_2 and B_3. B_1 is a bounding circle for the fractal if $\dfrac{R_{B_1} - F_1F_2}{R_{B_1} + F_1F_2} = s_2$

$$R_{B_1} = F_1F_2\frac{(1+s_2)}{(1-s_2)}$$

Note that we could have centred a bounding circle on F_2 instead of F_1, but the resulting circle would have a larger radius, as $s_1 \geq s_2$.

To apply this method to the general case of a fractal defined by more than two transformations, we treat the transformations pairwise, generating $n(n-1)/2$ circles, and use geometric techniques to yield a circle which contains all of these circles[9]. However, a tighter bounding circle can be derived using the method described in Section 4.

4. THE GENERAL CASE

For a fractal defined by n transformations T_i, $1 \leq i \leq n$, $n > 1$, we follow the approach of [7] and consider the effect of one of the transformations T_i on an arbitrary point X that lies on or inside a bounding circle B_i for T_i centred on another arbitrary point O. T_i maps O to the point O_i and X to the point X_i. We denote the radius of the bounding circle by R_{B_i} and note that OO_i is easily calculated from the coordinates of O.

B_i is a bounding circle for T_i if X_i lies inside B_i, i.e. if $OX_i \leq R_{B_i}$. Simple geometry tells us that $OX_i \leq OY + YX_i$ for any point Y, thus a sufficient condition to make B_i a bounding circle for T_i is $OO_i + O_iX_i \leq R_{B_i}$. Contractiveness tells us that $O_iX_i \leq s_iR_{B_i}$, thus a sufficient condition to make B_i a bounding circle for T_i is $OO_i + s_iR_{B_i} \leq R_{B_i}$. Solving for R_{B_i} gives

$$R_{B_i} \geq \frac{OO_i}{1-s_i}$$

Clearly, R_{B_i} is a minimum when the two sides are equal. Figure 3 illustrates the result.

Treating each transformation in this way gives us n concentric circles: we can derive a bounding circle B for the fractal simply by taking the maximum of their radii. The upper bound is therefore given by the formula

$$R_B = \max_{i=1}^{n} \frac{OO_i}{1-s_i}$$

The quality of the upper bound derived by this formula depends acutely on the position chosen for the centre, O. In the case of two transformations, taking O to be F_1 gives a tighter bound than in Section 3. In general, the optimum centre can be found easily using a searching algorithm: alternatively, it is possible to derive the optimum centre analytically[10].

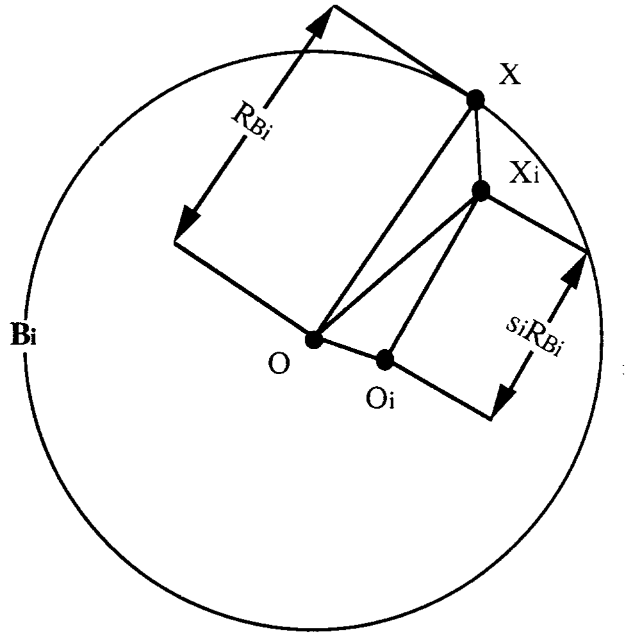

Figure 3 The circle B_i is a bounding circle for T_i if $R_{B_i} \geq \frac{OO_i}{1-s_i}$.

5. CONCLUSIONS

Plotting a fractal image from its transformations on the screen of a digital computer requires advance knowledge of a bounding area of the image. Previously, this bounding area has been determined by trial and error. We have derived a formula that expresses the radius of a bounding circle for a fractal in terms of the transformations representing the fractal. This will improve the effectiveness of algorithms that plot IFS fractal images.

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