EXTENDED PERMUTATION FILTERS AND THEIR APPLICATION TO EDGE ENHANCEMENT

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ABSTRACT

Extended permutation (EP) filters are defined and analyzed in this paper. In particular, we focus on extended permutation rank selection (EPRS) filters. These filters are constrained to output an order statistic from an extended observation vector. This extended vector includes N observation samples and K statistics that are functions of the observation samples. By selecting an appropriate extended observation space, we show that the EPRS filters can be designed to have excellent edge enhancement characteristics. Moreover, the EPRS filters can perform edge enhancement in the presence of noise making them a powerful filter class.

1. INTRODUCTION

Nonlinear filters have proven to be highly successful in many signal and image restoration applications. In particular, rank order based filters are well known for their ability to treat heavy tailed noise and nonstationary signals. With this type of data, linear filters tend to perform poorly. The first, and most well known, of these rank order based filters is the median filter. Many other more sophisticated rank order based filters have been proposed including *permutation* filters [1] and *rank conditioned rank selection* (RCRS) filters [4]. Rank order based filters have primarily been utilized as smoothing filters in restoration applications where a signal is corrupted by noise. In addition to performing smoothing operations, however, some rank order based filters can be effective in edge enhancement applications. These include the *comparison and selection* (CS) filter [6], the *lower-upper-middle* (LUM) filter [5] and the *weighted majority of samples with minimum range* (WMMR) filter [7]. The application of rank order based filters to edge enhancement has received more limited attention.

In this paper, we develop a filter class which provides a broad framework for formulating rank order based edge enhancing filters. These filters will be referred to as *extended permutation* (EP) filters and can be viewed as an extension of RCRS and permutation filters. The EP filters are are based upon a partitioning of the observation space using rank permutations of samples from an extended observation vector. This extended vector contains N observation samples and K statistics which are functions of the observation samples. A common filtering operation is defined for each partition, or ordering of the extended observation vector. While numerous filtering operations can be performed for each partition, we focus here on rank selection operations, and refer to the resulting filters as *extended permutation rank selection* (EPRS) filters.

The EPRS filters posses excellent noise smoothing capabilities as a result of their use of rank order information and their inclusion of RCRS and permutation filters as subsets. With well chosen statistics in the extended observation vector, the capabilities of EPRS filters can be made to include edge enhancement. We show that the sample mean is such a statistic and that the inclusion of the sample mean in the extended observation vector gives EPRS filters excellent edge enhancement properties. Finally, it is also shown that the new filters outperform previously defined rank order based edge enhancing filters in a Markov sequence restoration application.

This remainder of this paper is organized as follows. In Section 2, RCRS and permutation filters are defined since the EP filter development builds upon these definitions. The EPRS filters are defined in Section 3. Optimization is addressed and some edge enhancement properties are provided in Section 4. Simulation results are presented in Section 5. Finally, some conclusions are given in Section 6.

2. RCRS AND PERMUTATION FILTERS

Consider the d-dimensional discrete sequences $\{d(\mathbf{n})\}$ and $\{x(\mathbf{n})\}$, where the index $\mathbf{n} = [n_1, n_2, \ldots, n_d]$. Let these sequences represent the desired and corrupted versions of a signal respectively. Also, consider a window function that spans N samples and passes over the corrupted sequence in some predetermined fashion. At each location $\mathbf{n}$, the N observation samples spanned by the window can be indexed and written as a vector, yielding

$$\mathbf{x}(\mathbf{n}) = [x_1(\mathbf{n}), x_2(\mathbf{n}), \ldots, x_N(\mathbf{n})]. \quad (1)$$

The windowing and indexing of the observation sequence defines an ordering of the observed samples. Typically, this ordering is temporal for one-dimensional time sequences. Other orderings are possible, as are windows of

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higher dimension. An ordering that can be universally applied to the observed samples, regardless of signal dimension or window configuration, is rank ordering. The N observation samples ordered according to rank will be written as

$$x_{(1)}(\mathbf{n}) \leq x_{(2)}(\mathbf{n}) \leq \cdots \leq x_{(N)}(\mathbf{n}), \quad (2)$$

where $x_{(1)}(\mathbf{n}), x_{(2)}(\mathbf{n}), \ldots, x_{(N)}(\mathbf{n})$ are referred to as the order statistics of the observation.

The use of more than one ordering of the observed samples has proved advantages in many filtering problems [1, 4]. For instance, temporal correlations can be exploited if the temporal order of samples is known. In contrast, rank ordering allows for the effective rejection of outliers, since these samples are most often located in the extremes of the ranked set. By utilizing both orderings, results superior to the two marginal cases can be obtained. Thus, to relate the rank of a sample to its location within the window, we define $r_i(\mathbf{n})$ to be the rank of the sample in window location i.

The filtering, or estimation, problem can now be posed as follows. From the set of observation samples, we wish to form an estimate of the desired sample at location δ within the window. This estimate is denoted as $\hat{d}_\delta(\mathbf{n})$, where $1 \leq \delta \leq N$. In the remainder of the paper, the index $\mathbf{n}$ is assumed and is used explicitly only when necessary for clarity.

Consider the vector $\mathbf{r} = [r_{\gamma_1}, r_{\gamma_2}, \ldots, r_{\gamma_M}]$, which contains the ranks of M selected observation samples $x_{\gamma_1}, x_{\gamma_2}, \ldots, x_{\gamma_M}$, where $0 \leq M \leq N$. Let $\mathbf{r} \in \Omega_\mathbf{Z}$, where $\mathbf{z} = [M, N]$ and $\Omega_\mathbf{Z}$ contains all permutations of the N indices $1, 2, \ldots, N$ taken M at a time. The output of an M^{th}–order RCRS filter with window size N is given by

$$F_{RCRS}(\mathbf{x}) = x_{(\mathcal{S}(\mathbf{r}))}, \quad (3)$$

where $\mathcal{S}(\cdot)$ is said to be the selection rule and $\mathcal{S} : \Omega_\mathbf{Z} \mapsto \{1, 2, \ldots, N\}$ [4]. Thus, RCRS filter estimates are based on the temporal and rank order of M selected samples. If $M = N$, then $\mathbf{r}$ relates the temporal and rank order of each input sample and $\Omega_\mathbf{Z}$ is the group of permutations. In this case, the full permutation information is used and (3) defines the class of permutation filters [1]. Using $\mathbf{r}$ as the basis for rank selection has been shown to be effective for smoothing and frequency selection/rejection applications [1, 4]. However, using sample ranks alone is not effective for edge enhancement. This follows because for locally monotone sequences, the resulting rank vector is given by

$$\mathbf{r} = [1, 2, \ldots, N] \quad \text{or} \quad \mathbf{r} = [N, N-1, \ldots, 1] \quad (4)$$

and remains constant. Thus, for an RS filter with any rank based selection rule $\mathcal{S}(\cdot)$, the output along a monotone sequence will be a constant order statistic $x_{(k)}$, where $k \in \{1, 2, \ldots, N\}$. Thus, no edge gradient enhancement can be accomplished.

3. EXTENDED PERMUTATION FILTERS

The EP filters overcome the problem of edge enhancement by using the ranks of samples from an extended observation vector as the basis for output rank selection. Thus, define an extended observation vector as

$$\begin{aligned}\tilde{\mathbf{x}} &= [\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{N+K}] \quad (5)\\ &= [x_1, x_2, \ldots, x_N, F_1(\mathbf{x}), F_2(\mathbf{x}), \ldots, F_K(\mathbf{x})],\end{aligned}$$

where $F_i(\mathbf{x})$ is some function of the observation vector. This extended observation vector can be sorted as before, yielding

$$\tilde{x}_{(1)} \leq \tilde{x}_{(2)} \leq \cdots \leq \tilde{x}_{(N+K)}. \quad (6)$$

Also, let an extended rank vector be defined as

$$\tilde{\mathbf{r}} = [\tilde{r}_{\gamma_1}, \tilde{r}_{\gamma_2}, \ldots, \tilde{r}_{\gamma_M}, \tilde{r}_{\beta_1}, \tilde{r}_{\beta_2}, \ldots, \tilde{r}_{\beta_L}] \in \Omega_\mathbf{Z}, \quad (7)$$

where $1 \leq \gamma_i \leq N$, $N+1 \leq \beta_i \leq N+K$, and the limits on M and L are given by $0 \leq M \leq N$ and $0 \leq L \leq K$. The element $\tilde{r}_{\gamma_i}$ is the rank of $\tilde{x}_{\gamma_i} = x_{\gamma_i}$ in $\tilde{\mathbf{x}}$, and $\tilde{r}_{\beta_i}$ is the rank of $\tilde{x}_{\beta_i} = F_{\beta_i - N}(\mathbf{x})$. Thus, the extended rank vector lies in the extended rank permutation space which is denoted as $\Omega_\mathbf{Z}$, where $\mathbf{z} = [M, N, K, L]$.

Each unique extended rank vector $\tilde{\mathbf{r}} \in \Omega_\mathbf{Z}$ defines a distinct partition in the R^N observation space. EP filters are defined such that a common filtering operation is applied to each observation vector lying in a given partition. In the general case, the filtering operation performed is a function of the extended observation and can be either linear or nonlinear. For the EPRS filters considered here, the filtering method is restricted to an order statistic operation. That is, in each partition a specific order statistic from $\tilde{\mathbf{x}}$ is selected as the filter output. These filters are formally defined below.

Definition 3.1 *The output of an EPRS filter is given by*

$$F_{EPRS}(\mathbf{x}) = \tilde{x}_{(\mathcal{S}(\tilde{\mathbf{r}}))}, \quad (8)$$

where $\mathcal{S} : \Omega_\mathbf{Z} \mapsto \{1, 2, \ldots, N+K\}$.

The cardinality of the extended rank permutation space depends, in general, on the K functions. The cardinality is, however, bound above such that $|\Omega_\mathbf{Z}| \leq (N+K)!/(N-M+K-L)!$. Similarly, for each observed rank permutation, the number of possible unique EPRS filter outputs is less than or equal to $N+K$. Thus, denoting the class of EPRS filters as $\Psi_\mathbf{Z}$, the cardinality of the filter class is bound above by $|\Psi_\mathbf{Z}| \leq (N+K)^{|\Omega_\mathbf{Z}|}$. It can be shown that the CS, LUM and WMMR filters can be readily formulated as a subclass of EPRS filters [3] as can RCRS and permutation filters. Thus, the new filters comprise a broad class in which rank order based smoothers and sharpeners can be related.

Consider the case where $K = L = 1$ ($\beta_1 = N+1$) and $\tilde{x}_{N+1} = F_1(\mathbf{x})$ is an α–trimmed sample mean estimate. We show that this is an effective choice for edge enhancement applications. This follows because the rank of the mean provides information about where the observation window lies with respect to an edge midpoint. The rest of this paper will focus on this case. As the size of the extended vector $\tilde{\mathbf{x}}$ is now determined by M, we will refer to M as the order of the EPRS filter. It can be shown that the CS, LUM and WMMR filters can be readily formulated as a subclass of EPRS filters [3].

4. PROPERTIES AND OPTIMIZATION

Some edge enhancement properties of the EPRS filters are presented in this section. Proofs of these properties and additional properties can be found in [3]. First some definitions must be presented. To define convex and concave sequences, the first difference of samples is used. Let $\Delta(n)$ denote the first difference, $\Delta(n) = x(n) - x(n-1)$. Then, $\{x\}$ is convex (concave) if $\Delta(n) \geq \Delta(n-1)$ ($\Delta(n) < \Delta(n-1)$) for all n. Convex and concave sequences can be concatenated to form edges. An increasing sequence $\{x\}$, with first difference $\{\Delta\}$, contains a convex/concave edge with inflection point I if $\Delta(n) \geq \Delta(n-1)$ for $n \leq I$ and $\Delta(n) < \Delta(n-1)$ for $n > I$. Thus, $x(n)$ is convex for $n \leq I$ and concave for $n > I$.

The following property gives bounds on the rank of the α–trimmed mean for an observation window passing over such an edge. The proof of the property follows closely that in [6], with the span of the window modified by trimming off the α smallest and largest samples.

Property 4.1 (mean rank bounds) *For a window passing over an increasing convex/concave edge, there exists an integer m such that the rank of the α–trimmed mean, $F_1(\mathbf{x}(n))$, is bound below by $\tilde{r}_{\beta_1} \geq (N+1)/2$ for $n \leq m$ and bound above by $\tilde{r}_{\beta_1} < (N+1)/2$ for $n > m$. Moreover, the unique point m is within $N - 2(\alpha - 1)$ samples of the edge inflection point I, $|I - m| \leq N - 2(\alpha - 1)$.* □

The following property gives sufficient conditions on the selection rule $\mathcal{S}(\cdot)$ which results in an EPRS filter that reduces transition durations, or enhances edges.

Property 4.2 (Edge enhancement) *Let $F_{EPRS}(\cdot)$ be a window size N EPRS filter with $K = L = 1$ and $F_1(\cdot)$ the α–trimmed mean. Restrict the selection rule $\mathcal{S} : \Omega_{\mathbf{Z}} \mapsto \{1, 2, \ldots, N+1\}$ which defines $F_{EPRS}(\cdot)$ to be $\mathcal{S}(\tilde{\mathbf{r}}) = k_1$ when $\tilde{r}_{\beta_1} \geq (N+1)/2$ and $\mathcal{S}(\tilde{\mathbf{r}}) = N + 2 - k_2$ when $\tilde{r}_{\beta_1} < (N+1)/2$, for $1 \leq k_1, k_2 < (N+1)/2$. Take $\{x\}$ to be an increasing convex/concave edge sequence with inflection point I. For any two thresholds $T_1 < x(I) < T_2$ with transition duration $n_2 - n_1 > 2(N - 2(\alpha - 1))$, the transition duration of $F_{EPRS}(\{x\})$ is less than that of $\{x\}$. If $n_1 \leq I - (N + k_1 - 2(\alpha - 1))$ and $n_2 \geq I + (N + k_2 - 2(\alpha - 1))$, then $F_{EPRS}(\cdot)$ reduces the transition duration by $N - (k_1 + k_2) + 1$ samples.* □

This edge enhancing property is illustrated in Fig. 1. The figure contains a convex/concave edge filtered by an EPRS filter meeting the above edge enhancing conditions. The results shown are for a symmetric selection rule, $k_1 = k_2 = k$. For this simple selection rule, the EPRS filter is equivalent to the CS filter [6]. Such a basic selection rule can provide edge enhancement and allows for relatively simple analysis. However, in practice such a rule may perform poorly on signals with complex structures and edges. By considering the ranks of the α–trimmed mean and M selected observation samples, the EPRS filter can use more sophisticated selection rules. This allows the filter to adapt to a wider variety of signal structures and edges.

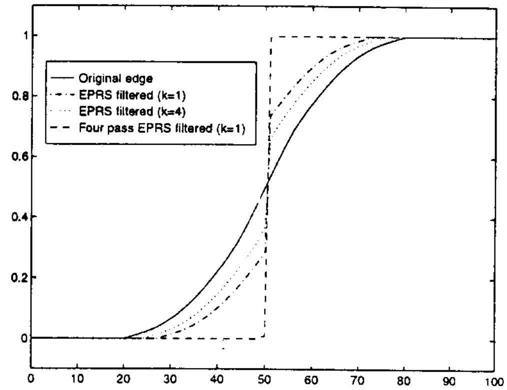

Figure 1: *A convex/concave edge filtered by an EPRS filter. The selection rule is chosen to be symmetric, $k_1 = k_2 = k$ and the window size is 15.*

Selecting a filter based solely on deterministic properties becomes less practical as the filter class size grows. A more practical solution is to optimize over the filter class utilizing training sequences that accurately account for the varied edge types present in the signal of interest. Provided that the training sequences $\{\mathbf{x}(\mathbf{n})\}$ and $\{\mathbf{d}(\mathbf{n})\}$ are available, the EPRS filter selection function $\mathcal{S}(\cdot)$ can be readily optimized under the the least L_η normed error (LNE) criteria. This procedure is described in [3] and closely follows that described in [1, 4]. The training procedure is deterministic and guarantees the globally optimal solution for the particular training data.

5. EXPERIMENTAL RESULTS

In this section, some experimental results are presented. The proposed filters could be useful in a number of image restoration applications where an image is blurred and corrupted by noise. However, to clearly illustrate the operation of the filters, a simple one-dimensional signal is used here. The signal is a five level Markov sequence. The data have been blurred with a 15 sample Gaussian point spread function and have been further corrupted with impulsive noise.

The input signal, corrupted signal, and signal restored using a EPRS filter are shown in Fig. 2. The EPRS filter is employing a window size of $N = 9$ and the partitions are determined from the rank of the center sample and the mean only ($M = 1$). The filter has been trained using a different signal and noise realization. Notice that the impulses are suppressed and the edge gradients have been fully restored in most places. The output of the filter using only the rank of the center sample (order one RCRS filter [4]) is shown in Fig. 3. Here the impulses are suppressed, but, the edge gradients have not been enhanced significantly.

A plot showing the mean absolute error (MAE) for some rank order based edge enhancers applied to the Markov signal is shown in Fig. 4. Different signals and noise realizations have been used for training and filter-

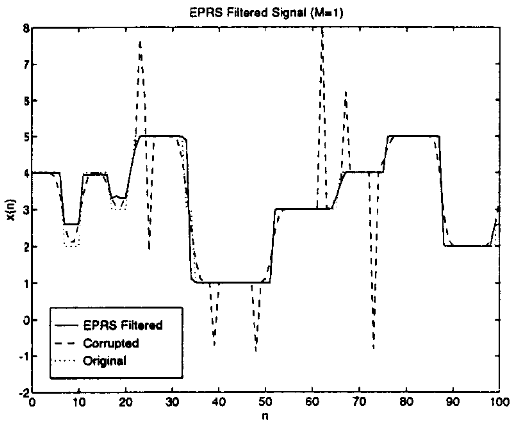

Figure 2: *Restoration of a blurred Markov sequence with impulsive noise (impulse probability of 0.1). A PRS filter is used with the rank of the center sample and the mean ($M = 1$ and $N = 9$).*

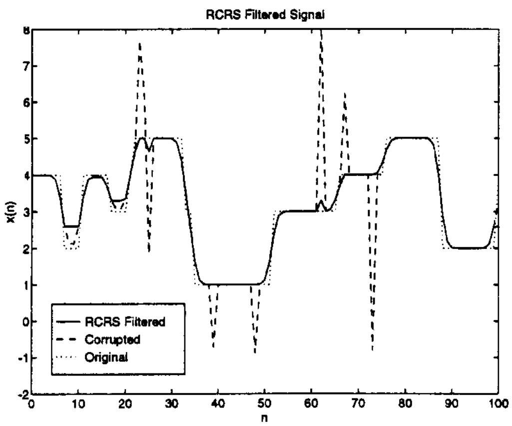

Figure 3: *Restoration of a blurred Markov sequence with impulsive noise. An order one RCRS filter is used with only the rank of the center sample ($M = 1$ and $N = 9$).*

ing. The EPRS filters yield the best results. Note that better performance can be obtained, at the expense of filter simplicity, by increasing M. Clearly the median has the worst performance since it does not provide any edge enhancement.

6. CONCLUSIONS

The EPRS filters use the rank of samples from an extended observation vector as the basis for selecting an order statistic to be the filter output. By simply including the mean in the extended observation vector, the filters exhibit excellent edge enhancement properties. The filters also have excellent noise suppression characteristics. Thus, edge enhancement can be performed in the presence of noise making the filter class very powerful.

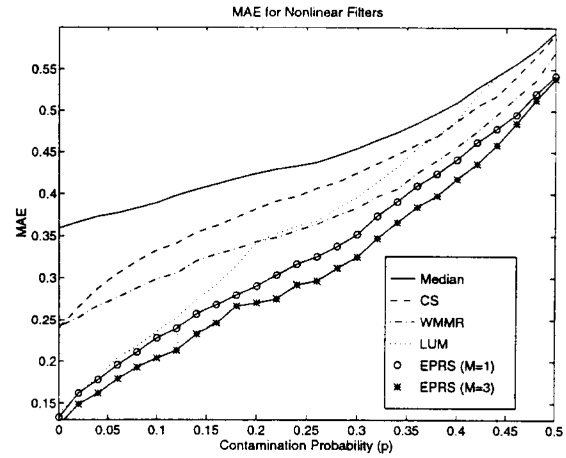

Figure 4: *MAE versus impulse probability for a number of rank order based edge enhancers. The filters are applied to the corrupted Markov signal.*

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