EXACT TIKHONOV REGULARISATION FOR THE LIMITED DATA COMPUTED TOMOGRAPHY PROBLEM

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ABSTRACT

We present a new variational approach to the problem of computed tomography reconstruction from sparse data. We use a Tikhonov regularisation (quite different from that of Louis(1985)) which deals without approximation with discrete or nonuniform grids.

1. INTRODUCTION

While convolution backprojection (CBP) is widely used in regular computed tomography [4] there seems to be no accepted algorithm for dealing with sparse data problems. Many existing procedures try to force the problem onto a uniform data CBP form [2] and so have discretisation errors and are noise sensitive.
The illconditioned nature of the sparse data reconstruction problem is well studied [4], and requires a regularisation to deliver a stable solution. A recent Tikhonov based approach is due to [5] however they do not explicitly recognise data discreteness and they regularise the sinogram rather than the underlying object. The approach of [3] deals directly with the underlying object but does not deal with data discreteness.

In this work we develop a Tikhonov regularisation solution which unlike those above explictly deals with the data discreteness and we regularise the item of fundamental interest, the underlying object, and not the intermediate sinogram quantity.

2. REGULARISATION

Suppose we have projection data of the following form

$$y_{uj} = P_{\theta_u}(t_j) + n_{uj} \quad u = 1...m, j = 1..n \qquad (1)$$

where there are one m angles and n observations per angle. $P_\theta(t)$ is the Radon transform

$$P_\theta(t) = \int f(\underline{x})\delta(t - \underline{x}.\underline{e}_\theta)d\underline{x} \qquad (2)$$

$[e_\theta = (\cos\theta, sin\theta)]$ and is the projection of the density $f(\underline{x})$ along lines $\underline{x}.\underline{e}_\theta = t$ at angle θ to the x, y co-ordinate system. Also n_{uj} is a white noise.

The aim is to reconstruct $f(x)$ from the data $\{y_{uj}\}$. The difficulty in such an ill-conditioned inversion is well studied [4]. Here we pursue a Tikhonov regularisation apoproach which is quite different from those of [3], [5]. In particular in [3] the discreteness of the data domain is not recognised. On the one hand in reconstructing the image $f(\underline{x})$ one wants to retain some fidelity to the data by keeping J_d small where

$$J_d = \Sigma_{u=1}^m \Sigma_{j=1}^n (y_{uj} - P_{\theta u}(t_j))^2 \qquad (3)$$

But to avoid obtaining too noisy a reconstruction that a very small J_d would entail one also tries to enforce some smoothness by keeping J_c small

$$J_c(f) = \int_\Omega (f_{xx}^2 + 2f_{xy}^2 + f_{yy}^2)d\underline{x} \qquad (4)$$

where $f_{xx} = \partial^2 f/\partial x^2$ etc; Ω is the region of support of $f(\underline{x})$ (which we take to be a disc). This functional measures the bending energy of a thin plate [1, section IV.9] and can be interpreted as a smoothness measure and is the basis of the Thin plate Smoothing spline in function estimation [6].

To trade off these two conflicting criteria one is led to a regularisation index of the form $J = J_d + \alpha J_c$ where α is a penalty parameter to be chosen. We minimise J with respect to $f(\underline{x})$ subject to the constraint (2).

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The resultant continuous-discrete variational problem is nonstandard and leads to the following solution

$$\begin{aligned} f(\underline{x}) &= \Sigma_u \Sigma_j \lambda_{uj} g_{uj}(\underline{x}) + \Sigma_1^3 \phi_\nu(\underline{x}) d_\nu \\ (\phi_1(\underline{x}), \phi_2(\underline{x}), \phi_3(\underline{x})) &= (1, x_1, x_2) \\ g_{uj}(\underline{x}) &= \int G(\underline{x}; \underline{y}) \delta(t_j - \underline{y}.\underline{e}_{\theta_u}) d\underline{y} \end{aligned}$$

where $G(\underline{x}; \underline{y})$ is a certain Green's function for the biharmonic operator ∇^4 on the unit disc and $\{\lambda_{uj}\}$ is obtained from the following matrix equations.

$$\begin{aligned} (\underline{Q} + \alpha \underline{I})\underline{\lambda} + \underline{Nd} &= y \\ \underline{N}^T \underline{\lambda} &= \underline{0} \end{aligned}$$

$$\begin{aligned} \underline{y} &= [\ldots y_{u1} y_{u2} \ldots y_{un} \ldots]^T \\ \underline{d} &= (d_1 d_2 d_3)^T \\ \underline{\lambda} &= [\ldots \lambda_{u1} \lambda_{u2} \ldots \lambda_{un} \ldots]^T \end{aligned}$$

$\underline{Q}$ is a matrix made from the tensor

$$\begin{aligned} Q_{uj,rs} &= Q(\theta_u, t_j; \theta_r, t_s) \\ Q(\theta, t; \phi, \tau) &= \int_\Omega \int_\Omega \delta(t - \underline{e}_\theta.\underline{x}) G(\underline{x}; y) \delta(\tau - \underline{e}_\phi.\underline{y}) d\underline{x} d\underline{y} \end{aligned}$$

$\underline{N}$ is a matrix assembled from the tensor

$$\begin{aligned} N_{uj\nu} &= N_\nu(\theta_u, t_j) \\ N_\nu(\theta, t) &= \int_\Omega \delta(t - \underline{e}_\theta.\underline{x}) \phi_\nu(\underline{x}) d\underline{x} \end{aligned}$$

Further details of the computations involved, of the choice of α using cross validation and some examples will be given at the conference.

3. REFERENCES

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