WAVELET PACKET BASED OPTIMAL SUBBAND CODER.

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ABSTRACT

In this work, we propose an algorithm to use in designing a subband coder (SBC) constructed by wavelet packet, to achieve minimum distortion for given bit budget and implementation complexity. We map the QMF tree structures onto a binary tree, then formulate the task as an optimization problem including coding bit and implementation complexity constraints. The problem is dissected into two phases. First, we derive the optimal bit allocation strategy which covers the entire range of bit rate, and second, we search for the optimal subband decomposition by using a fast dynamic program.

1. INTRODUCTION

Since a wavelet packet is useful in designing a subband coder (SBC) because it provides flexible subband decomposition to meet a signal's spectral behaviors [1]. The performance of an SBC improves as the coding bit rate and number of subbands increase. However, the higher coding bit rates result in the higher transmission costs, and high number of subbands must lead to more implementation complexities. Because of these trade-offs, it is necessary to find the optimal bit allocation for a given coding bit budget, as well as the optimal subband decomposition for a limited implementation complexity. These two trade-offs are also closely coupled, so the determination of bit allocation depends on the determination of subband decomposition, and vice versa. In this work, a new approach to designing a wavelet packet based SBC is presented. Then the proposed method is applied to low bit image compression with the use of vector quantization (VQ). In the first phase, we provide a strategy for designing optimal subband quantizers for a certain subband decomposition, and then in the second phase, we search for the optimal subband decomposition for a given optimal quantizor set. Therefore, unlike previous studies [1], [2], our design method simultaneously provides the optimal quantizer scheme and the optimal subband decomposition scheme.

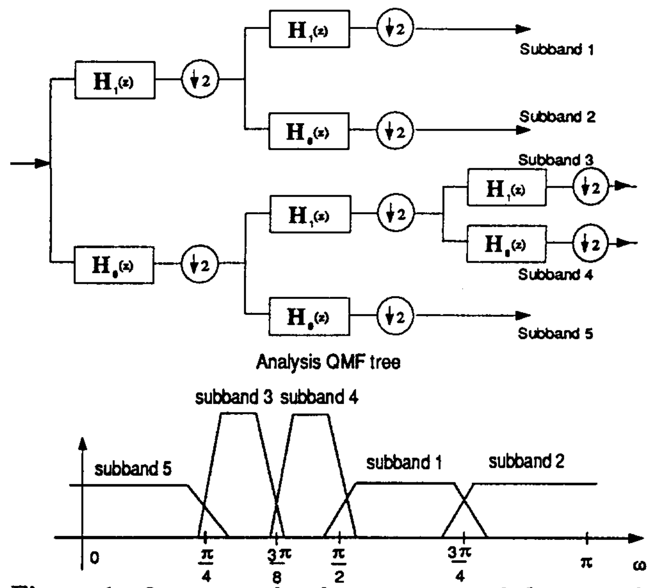

Figure 1: One example of tree structured QMFs and its nonuniform subband decomposition.

2. WAVELET PACKET AND TREE STRUCTURED QMFS

The tree structured connections of quadrature mirror filters (QMF) realize nonuniform subband decompositions, as shown in Figure 1 [3]. The family of wavelet bases constructing such nonuniform filter banks is referred to as a wavelet packet. One advantage of wavelet packet lies in flexible adjustments of the bandwidth in respond to a given frequency location.

In separable space, a two dimensional wavelet basis can be the product of vertical and horizontal wavelet bases. Since the separability makes it possible to convert 2-D QMF into a serial connection of horizontal and vertical QMFs, row and column operations alternate. Thus, the resolution of a 2-D subspace is the product of row and column resolution. In this work, to make full use of this separability, row and column resolutions may be different, and the 2-D subspace shape is not necessarily square.

3. DEFINITION AND NOTATION

To deal with the QMF trees, the followin definitions and notations are needed.

- S : Tree mapping the structure of QMF tree. Node 1 is the root node. The branch from node i to node $2i$ includes lowpass filter bank, and the branch from node i to node $(2i+1)$ includes highpass filter bank.
- $L(S)$: Leaf node set of tree S.
- d_i: Depth of a node i, or $d_i = \lfloor \log_2 i \rfloor - 1$. The subband size corresponding to node i is $\pi/2^{d_i}$.
- $Q(S)$: Quantizer set for the subband coder S.
- $E(S, Q(S))$: Lower bound of reconstruction error variance, or mean square rate distortion evoked by the subband coder S with quantizer set $Q(S)$.

4. PROBLEM DEFINITION

If a subband signal whose bandwidth is $\pi/2^{d_i}$ is encoded by R_i bits/sample, the transmission rate of the subband signal is $2 * \frac{\pi}{2^{d_i}} * R_i$. Since the sum of the subband transmission rates is equal to the transmission rate of the coder, with total R bits/sample, we obtain $R = \sum_{i \in L(S)} \frac{R_i}{2^{d_i}}$. In addition, the cost for implementing the subband i can be calculated as the implementation complexity value w_i. So, the implementation cost will be $\mathbf{W}(S) = \sum_{i \in L(S)} w_i \leq W$

Then, the task can be defined as an optimization problem like the following

$$E(S^*, Q^*(S)) = \min_{(S, Q(S))} E(S, Q(S)) \qquad (1)$$

$$\text{Bit constraint :} \quad \mathbf{R}(S) = \sum_{i \in L(S)} \frac{R_i}{2^{d_i}} \leq R$$

$$\text{Complexity constraint :} \quad \mathbf{W}(S) = \sum_{i \in L(S)} w_i \leq W$$

5. RECONSTRUCTION ERROR ANALYSIS

Without loss of generality, we can assume that the subband size of a channel is $\pi/M = \pi/2^{d_i}$. The channel filter banks denoted as $h^i(n)$ and $f^i(n)$ are made of d_i connections of QMFs. Since quantization noise $q_i(n)$ is uncorrelated with the subband signal, the channel error $e_i(n)$ is produced by only $q_i(n)$, and uncorrelated with the channel reconstruction signal. Then, the reconstruction error is

$$e(n) = \sum_{i \in L(S)} e_i(n) = \sum_{i \in L(S)} \left\{ \sum_r q_i(r) f^i(n - rM) \right\}$$

Theorem 1 *The reconstruction error variance of $e_i(n)$ is calculated as*

$$E\{|e_i(n)|^2\} = \frac{1}{M} E\{|q_i|^2\} \sum_m |f^i(m)|^2$$

Proof: Since $e_i(n)$ is a cyclostationary process with the period M, the average autocorrelation of $e_i(n)$ is computed by taking a time average with respect to cyclostationary period M. Then

$\bar{R}_i(n) = \frac{1}{M} \sum_{l=0}^{M-1} R_i(l, n+l)$

$= \frac{1}{M} \sum_{l=0}^{M-1} \sum_r \sum_s f^i(l - rM) f^i(l + n - sM) R_q(r - s)$

Since $R_q(r-s) = \sigma_{q_i}^2 \cdot \delta(r-s)$,

$= \frac{\sigma_{q_i}^2}{M} \sum_{l=0}^{M-1} \sum_r f^i(l - rM) f^i(l + n - rM)$

Thus, the reconstruction error variance is

$$\begin{aligned} E\{|e_i(n)|^2\} &= \bar{R}_i(0) = \frac{\sigma_{q_i}^2}{M} \sum_{l=0}^{M-1} \sum_r \left|f^i(l - rM)\right|^2 \\ &= \frac{\sigma_{q_i}^2}{M} \sum_m \left|f^i(m)\right|^2 \end{aligned} \qquad \square$$

If the Shannon lower bound is used, the lower bound of reconstruction error variance of the SBC with subband decomposition S is expressed as

$$E(S, Q(S)) = \sum_{i \in L(S)} \frac{2^{-2R_i} \beta_i \sigma_{v_i}^2}{2^{d_i}} \qquad (2)$$

where R_i is the bit number of quantizer q_i, $\beta_i = \sum_m \left|f^i(m)\right|^2$ which is filter energy, and $\sigma_{v_i}^2$ is the signal variance of subband i.

6. PHASE I: OPTIMAL BIT ALLOCATION

Since $Q(S)$ is functions of allocated bits, the optimization problem at this stage is

$$\min_{Q(S)} E(S, Q(S)) = \min_{R_i} \left\{ \sum_{i \in L(S)} \frac{\epsilon^2 2^{-2R_i} \beta_i \sigma_{v_i}^2}{2^{d_i}} \right\} \qquad (3)$$

$$\text{such that} \quad \sum_{i \in L(S)} \frac{R_i}{2^{d_i}} \leq R, \quad R_i \geq 0$$

The following theorem solves (3).

Theorem 2 *Define the subsets $L'(S)$ and $L''(S)$ for a distortion threshold θ to be*

$$L'(S) = \{\text{leaf node } i \mid \beta_i\sigma_{v_i}^2 \geq \theta,\ i \in S\}$$
$$L''(S) = \{\text{leaf node } i \mid \beta_i\sigma_{v_i}^2 < \theta,\ i \in S\}$$

Then, θ is explicitly solved as

$$\theta = 2^{-2R/\alpha(S)} \left[\prod_{i\in L'(\theta,S)} (\beta_i\sigma_{v_i}^2)^{1/2^{d_i}} \right]^{1/\alpha(S)} \quad (4)$$

where $\alpha(S) = \sum_{i\in L'(S)} \frac{1}{2^{d_i}}$
So, the optimal bit allocation is given by

$$R_i = \begin{cases} 0 & \text{if } \beta_i\sigma_{v_i}^2 \leq \theta \\ \frac{1}{2}\log_2\left(\frac{\beta_i\sigma_{v_i}^2}{\theta}\right) & \text{if } \beta_i\sigma_{v_i}^2 \geq \theta \end{cases}$$

and the minimum distortion becomes

$$\min_{Q(S)} E(S, Q(S)) = \epsilon^2 \sum_{i\in L(S)} \min\left(\frac{\theta}{2^{d_i}}, \frac{\beta_i\sigma_{v_i}^2}{2^{d_i}}\right) \quad (5)$$

Proof: The unconstrained problem equivalent to (3) is

$$\min_\lambda \left\{ \sum_{i\in L(S)} \min_{R_i\geq 0} \left\{ \lambda\frac{R_i}{2^{d_i}} + \frac{\beta_i\sigma_{v_i}^2}{2^{d_i}}\rho(R_i) \right\} - \lambda R \right\} \quad (6)$$

where $\rho(R_i) = \epsilon^2 2^{-2R_i}$.

Since $\rho(R_i)$ is decreasing and convex,

$$\min_{R_i\geq 0} \left\{ \lambda\frac{R_i}{2^{d_i}} + \frac{\beta_i\sigma_{v_i}^2}{2^{d_i}}\rho(R_i) \right\}$$

leads to

$$\begin{array}{ll} R_i = 0 & \text{if } \lambda \geq -\beta_i\sigma_{v_i}^2\rho'(0) \\ \lambda + \beta_i\sigma_{v_i}^2\rho'(R_i) & \text{if } 0 < \lambda < -\beta_i\sigma_{v_i}^2\rho'(0) \end{array} \quad (7)$$

Let $\theta = \lambda/(2\epsilon^2 \ln 2)$. Then, the optimal bit allocations are given from (7) such as

$$R_i = \begin{cases} 0 & \text{if } \beta_i\sigma_{v_i}^2 \leq \theta \\ \frac{1}{2}\log_2\left(\frac{\beta_i\sigma_{v_i}^2}{\theta}\right) & \text{if } \beta_i\sigma_{v_i}^2 \geq \theta \end{cases} \quad (8)$$

By substituting (8) into $\sum_{i\in L(S)} R_i/2^{d_i} = 2R$, the parameter θ can be found as the solution of

$$\mathrm{S}(\theta) = \sum_{i\in L'(\theta,S)} \frac{1}{2^{d_i}} \log_2\left(\frac{\beta_i\sigma_{v_i}^2}{\theta}\right) = 2R \quad (9)$$

$S(\theta)$ is strictly decreasing on $\theta = [0, \max\beta_i\sigma_{v_i}^2]$, and $S(0) = \infty$, $S(\max\beta_i\sigma_{v_i}^2) = 0$. Therefore, θ has the unique positive solution, and is solved as

$$\theta = 2^{-2R/\alpha(S)} \left[\prod_{i\in L'(S)} (\beta_i\sigma_{v_i}^2)^{1/2^{d_i}} \right]^{1/\alpha(S)} \quad (10)$$
where $\alpha(S) = \sum_{i\in L'(S)} \frac{1}{2^{d_i}}$

By putting (8) into (3), the minimum distortion is obtained as

$$\min_{Q(S)} E(S, Q(S)) = \epsilon^2 \sum_{i\in L(S)} \min\left(\frac{\theta^*}{2^{d_i}}, \frac{\beta_i\sigma_{v_i}^2}{2^{d_i}}\right)$$

Q.E.D

For the case of high bit rate, since $L'(\theta, S) = L(S)$, and $\sum_{i\in L'(\theta,S)} \frac{1}{2^{d_i}} = 1$, θ becomes

$$\theta = 2^{-2R} \left\{ \prod_{i\in L(S)} (\beta_i\sigma_{v_i}^2)^{1/2^{d_i}} \right\} \quad (11)$$

7. PHASE II: OPTIMAL SUBBAND DECOMPOSTION

The task in this phase is to find the optimal subband decomposition S^* for a given implementation complexity W, with the assumption that the subband quantizor set $Q^*(S)$ is optimally decided. From (11) and (5),

$$E(S, Q^*(S)) \geq \epsilon^2 2^{-2R} \prod_{i\in L(S)} (\beta_i\sigma_{v_i}^2)^{1/2^{d_i}} \quad (12)$$

After taking the logarithm of (12), the optimization problem at this stage is stated as

$$\min_S\ O(S) = \min_S \left\{ \sum_{i\in L(S)} \frac{1}{2^{d_i}} \log\beta_i\sigma_{v_i}^2 \right\} \quad (13)$$

$$\text{such that} \quad \sum_{i\in L(S)} w_i \leq W$$

For most real signals, the reconstruction error decreases as the number of subbands is increased. So, if a cost function $W(S)$ is increasing for the number of subbands, $O(S)$ is a decreasing function. With this fact, it is proved that the constrained optimization problem (13) can be converted to an unconstrained optimization problem with the Lagrange multiplier.

Theorem 3 *If a cost function $W(S)$ is a increasing function for the number of subbands, $|L(S)|$, the unconstrained problem for fixed $\lambda > 0$,*

$$\min_S \{O(S) + \lambda W(S)\} \quad (14)$$

solves the constrained optimization problem (13) when $W(S^*) = W_c$.

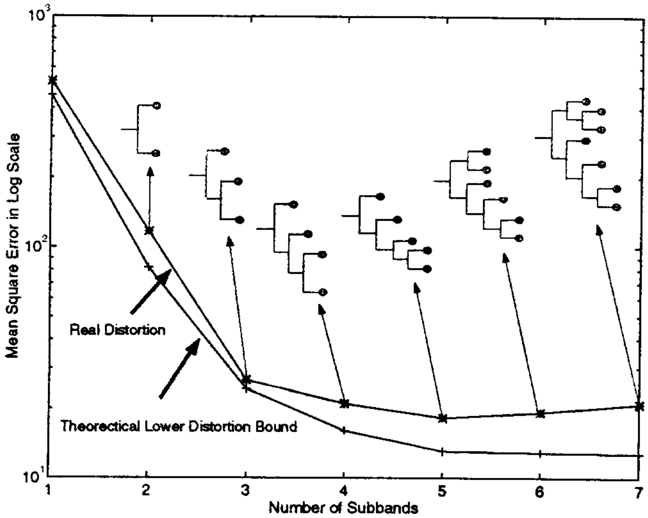

Figure 2: Optimal Subband Decomposition for Lena and it's distortion curve for 1 bit/pixel.

Proof: Let S^* the solution of (14), and then for any subband structures satisfying $W(S) \leq W_c$,

$$\begin{aligned} & O(S^*) + \lambda W(S^*) && \leq O(S) + \lambda W(S) \\ \text{Or,} \quad & O(S^*) - O(S) && \leq \lambda (W(S) - W(S^*)) \\ \text{And,} \quad & O(S^*) - O(S) && \leq \lambda (W(S) - W_c)) \end{aligned}$$

Since $O(S)$ is strictly decreasing for $W(S)$, and $\lambda > 0$,

$$O(S^*) \leq O(S) \quad \text{for} \quad W(S) \geq W(S^*) = W_c$$

Therefore, if the solution S^* of an unconstrained problem happens to be $W(S^*) = W_c$, the unconstrained problem (14) is identical to the constrained problem (13). *Q.E.D*

Theorem 3 implies that as λ sweeps over positive numbers, all operating points of $(O(\lambda), W(\lambda))$ are created. The operating points draw a convex hull. The essence of this algorithm is that the Lagrangian costs of parent node and child nodes are compared at each node, and if the Lagrangian cost by child nodes is more expensive, then the subtree hanging on the parent node is pruned. The tree of the surviving paths is optimal for a fixed $\lambda > 0$, therefore, the algorithm constructs only the optimal tree for a given $\lambda > 0$.

8. APPLICATION TO IMAGE COMPRESSION

Figure 2 shows the optimal subband decompositions obtained from the proposed algorithm and the real distortion curve for 1 bit/pixel compression. Unlike conventional R-D curves, the distortions are plotted versus the number of subbands. The real distortion is compatible with the theoretical lower bound distortion curve.

Figure 3: 0.1 bit/pixel without entropy coding, 5 subbands. PSNR is 27.53 dB

So, it is verified that the proposed algorithm can predict the optimal performance of an SBC, as well as provide the optimal designs for given coding bit budgets and implementation complexities. As an example of low bit compression, Figure 3 shows a 0.1 bit/pixel compression of the 8 bit/pixel 400x400 "Lena", without entropy coding. The number of subbands is 5, and follows the subband decomposition appearing at Figure 2. Compared as to bit rate, the picture does not suffer severe degradation and a blocking effect. Therefore, the proposed algorithm obtains competitive performance with lower bit rates and fewer implementation complexities.

9. REFERENCES

[1] R. Coifman, Y. Meyer, S. Quake, and V. Wickerhauser, "Signal processing and compression with wave packets," Technical report, Numerical Algorithms Research Group, New Haven, CT:Yale University, 1990.

[2] K. Ramchandran and M. Vetterli, "Best wavelet packet bases in a rate-distortion sense," *IEEE trans. image processing.*, vol. 2, no. 2, pp. 160–174, April 1993.

[3] P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice Hall, 1993.