ON THE APPLICABILITY OF 2-D DAMPED EXPONENTIAL MODELS TO SYNTHETIC APERTURE RADAR

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ABSTRACT

This paper examines the modeling of synthetic aperture radar (SAR) phase histories with 2-D damped exponential models of low order. The use of a low order model is warranted when the radar returns are attributable to a small number of point scatterers. In this paper we show that the fit of the widely used damped exponential model is highly dependent on the image scene. Specifically, current high resolution methods have limited applicability due to mismatch between the assumed model and observed data.

1. Damped Exponential Model

The 2-D damped exponential model has been successfully applied to array processing and radar problems [8]. In a slight extension of this model we consider damped exponentials in the presence of unmodeled dynamics and noise. The data are of the form $w(m_1, m_2)$, where $0 \leq m_1 \leq M_1 - 1$, and $0 \leq m_2 \leq M_2 - 1$. They have a 2-D harmonic structure:

$$w(m_1, m_2) = \sum_{i=1}^{p} s_i \lambda_i^{m_1} \gamma_i^{m_2} + n(m_1, m_2). \quad (1)$$

Here p is the number of harmonics (scatters) and $n(m_1, m_2)$ is due to unmodeled dynamics and noise. Each 2-D harmonic is characterized by two complex parameters, λ_i for the first dimension, and γ_i, for the second dimension. λ_i and γ_i are not restricted to lie on the unit circle, thus allowing for damped exponential signals. The complex number s_i is the amplitude of the i^{th} 2-D harmonic signal. For this paper the parameters of the model are treated as deterministic. Representing equation 1 in matrix form we have W, an $M_1 \times M_2$-dimensional matrix whose $(m_1, m_2)^{th}$ element is $w(m_1, m_2)$. Then

$$W = GSH^T + N, \quad (2)$$

where

$$G = [\,\boldsymbol{h}_1 \quad \boldsymbol{h}_2 \quad \ldots \quad \boldsymbol{h}_p\,], \boldsymbol{h}_i = \Psi_N(\lambda_i),$$

$$S = \text{diag}\{[s_1 \; s_2 \; \cdots \; s_p]\},$$

$$H = [\,\boldsymbol{g}_1 \quad \boldsymbol{g}_2 \quad \ldots \quad \boldsymbol{g}_p\,], \boldsymbol{g}_i = \Psi_M(\gamma_i),$$

and

$$\Psi_d(z) = [\,1 \quad z \quad z^2 \quad \ldots \quad z^{d-1}\,]^T.$$

N is the $M_1 \times M_2$ dimensional matrix whose $(m_1, m_2)^{th}$ element is $n(m_1, m_2)$.

We can apply this model to synthetic aperture radar (SAR) problems where the $M_1 \times M_2$ focused image, X, and phase history, W, are represented as a discrete Fourier transform (DFT) pair [6] by

$$X = D_{M_1} W D_{M_2}^T, \quad (3)$$

where D_M is the normalized DFT matrix with $(k, l)^{th}$ element $e^{-j\frac{2\pi}{M}kl}/\sqrt{M}$, $0 \leq k, l \leq M - 1$

Electromagnetic phenomenology often dictates that the phase histories are comprised of a small number of damped exponentials due to point scatterers [2, 4, 7]. To bound the performance of parameter estimation schemes one might consider other low rank models whose errors are easily calculated. To this end we consider both a generalization of and a restriction on the model of 2.

2. Error Bounds

A measure of the error between the phase histories W, and the estimated model, $\hat{W}$, is given by

$$e \equiv ||E||_F \equiv ||W - \hat{W}||_F. \quad (4)$$

Consider the nonlinear least squares (NLS) problem of minimizing equation 4 with respect to $\hat{W}$. Its solution is equivalent to the deterministic maximum

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likelihood (ML) estimate in the case where N is due only to white Gaussian noise. Even in the case of ARMA noise minimization of 4 yields an asymptotically statistically efficient estimate for undamped modes [10]. In the following section we show how the minimum error damped exponential solution to equation 4 is bounded.

2.1. Low Rank Approximation

A general low rank decomposition of the $M_1 \times M_2$ dimensional matrix, W, is

$$W = W_p + E, \quad (5)$$

where W_p is rank-p and E is a residual matrix. Of course the best approximation of order p is found by using by the SVD of $W : W = U\Sigma V^*$. Here the approximated matrix, $\hat{W}$, can be assembled from rank-p versions of the SVD component matrices as

$$\hat{W} = U(1:p,:)\Sigma(1:p,1:p)V(1:p,:)^* \quad (6)$$

where Σ contains the p largest singular values, the p columns of U are their left singular vectors, and the p columns of V^* are their right singular vectors. Thus the rank-p SVD approximation provides a lower bound on the error of all other rank-p approximations in the form of equation 5. The error attained by the rank-p SVD approximation is given by

$$e_{SVD} = ||E_{SVD}||_F = \sqrt{\sum_{i=p+1}^{\min(M_1,M_2)} \sigma_i^2} \geq \sigma_{p+1}. \quad (7)$$

2.2. Exponential Approximation

Now consider the rank-p model of equation 2. Since this is a restriction on equation 5 this limits the accuracy of our rank-p approximation. Its error is no better than that of equation 7. Thus $e_{damp} \geq e_{SVD}$.

2.3. DFT Frequency Approximation

Now consider a further restriction of equation 2 where λ_i, and γ_i can take on only discrete values. Specifically, $\lambda_i \in \{e^{j\frac{2\pi}{M_1}k}\}_{k=0}^{M_1-1}$ and $\gamma_i \in \{e^{j\frac{2\pi}{M_2}l}\}_{l=0}^{M_2-1}$. Observe from equations 3 and 4 that the minimum is attained by selecting the p values of X with largest moduli. As before this represents a restriction to the damped exponential model. So it is not surprising that corresponding errors satisfy $e_{DFT} \geq e_{damp} \geq e_{SVD}$.

3. Model Fitting

A set of SAR data was collected to determine the fit of the low rank approximations discussed in section 2. The SAR data was produced by XPATCH

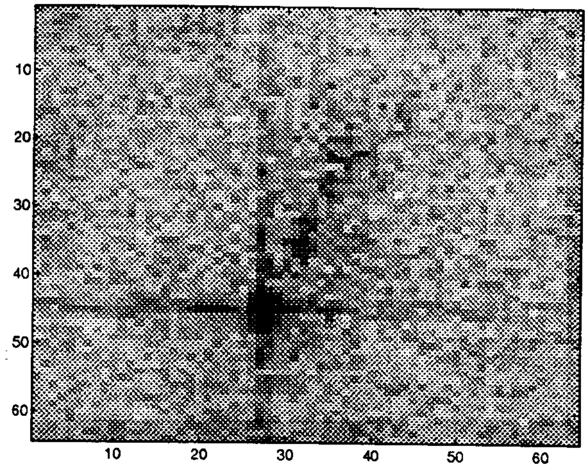

Figure 1: Firetruck

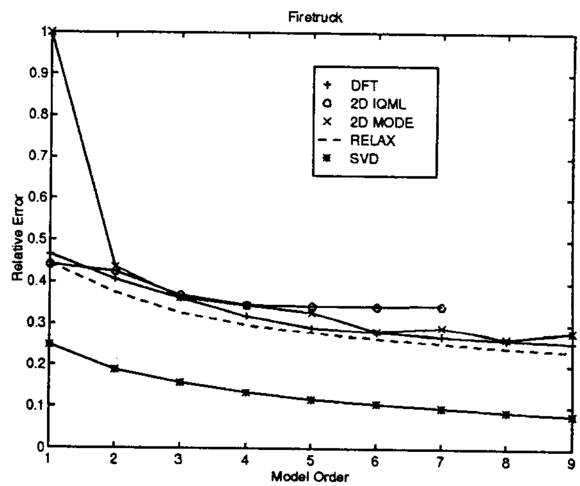

Figure 2: Representation Error vs. Model Order

software [1]. Normalized error between an estimated model and the phase history was calculated as

$$\bar{e} = \frac{||W - \hat{W}||_F}{||W||_F} \quad (8)$$

The error bounds given by the SVD and the DFT were established for this data. These represent lower and upper bounds, respectively, between which we would expect any reasonable approximation method to perform.

Two parametric methods that estimate the damped 2-D exponential model, 2-D IQML [3], and 2-D MODE [5], and a Fourier based method, RELAX were then tested.

The RELAX algorithm is a robust algorithm for estimating 2-D undamped sinusoidal parameters by minimizing the cost function in (4) with a relaxation process.

The steps of the RELAX algorithm may be briefly summarized as follows:

Step 1: Assume that there is one 2-D complex sinusoid in the SAR phase history data. The frequency estimate of the sinusoid is obtained as the location of the dominant peak of the 2-D periodogram, which can be efficiently computed by using FFT with the data matrix padded with zeros. (Note that padding with zeros is necessary to determine the frequency estimate with high accuracy.) Then the complex amplitude of sinusoid is computed from the complex height of the peak of the 2-D FFT of the data matrix.

Step 2: Assume that there are two 2-D complex sinusoids in the SAR phase history data. The corrected SAR phase history data is obtained by subtracting from the SAR phase history data the first sinusoid estimated in Step 1. The parameters of the second sinusoid are obtained similarly from the corrected SAR phase history data as described in Step 1. Then the parameters of the first sinusoid are redetermined from the data obtained by subtracting from the SAR phase history data the estimated second sinusoid. This process is iterated until the NLS cost function is minimized.

Step 3: Assume that there are three 2-D complex sinusoids in the SAR phase history data. The corrected SAR phase history data is obtained by subtracting from the SAR phase history data the first and second sinusoids estimated in Step 2. The parameters of the third sinusoid are obtained similarly from the corrected SAR phase history data as described in Step 1. Then the parameters of the first sinusoid are redetermined from the data obtained by subtracting from the SAR phase history data the estimated second and third sinusoids. This process is iterated until the NLS cost function is minimized.

The remaining steps are similar and the RELAX algorithm stops when the desired or estimated number of sinusoids or dominant scatterers of the target is reached.

Results for images shown in Figures 1 and 3 are plotted in Figures 2 and 4. The locations of the scatters identified by each algorithm are shown in Figures 5, 6, and 7.

4. Discussion

If the correct model order is used the SVD error provides a lower bound on the error in fitting the exponential model. Observe in Figures 2 and 4 that none of the parametric methods approached this bound. Either the data contains unmodeled signal (non-exponential), or the algorithms are producing poor estimates, or both. Additionally, 2-D IQML,

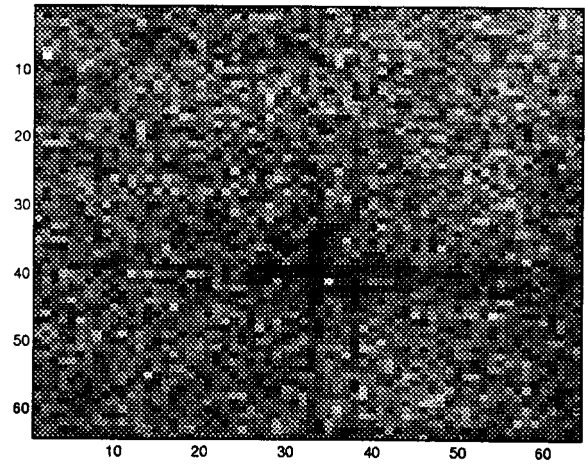

Figure 3: T-72

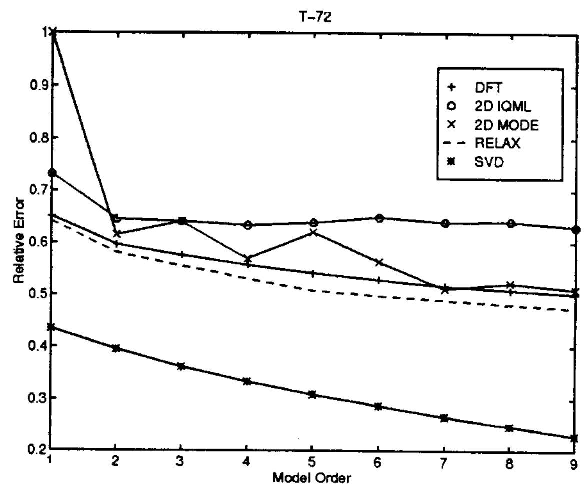

Figure 4: Representation Error vs. Model Order

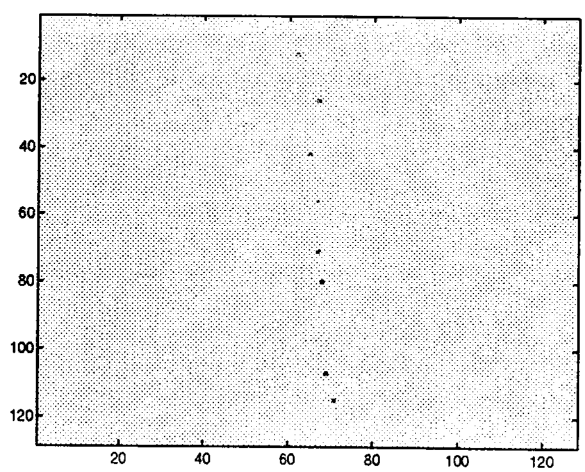

Figure 5: T-72 Scatters with 2-D IQML

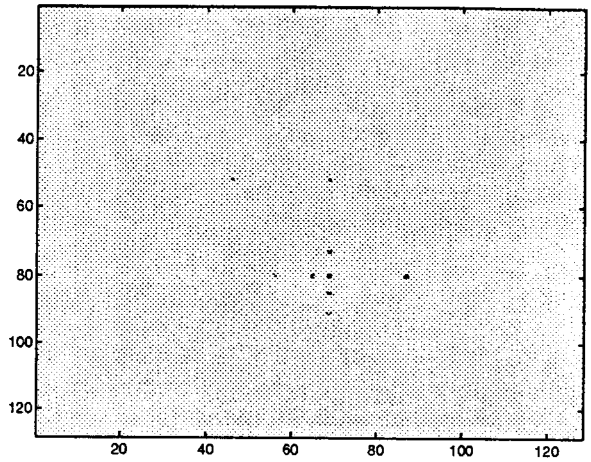

Figure 6: T-72 Scatters with 2-D MODE

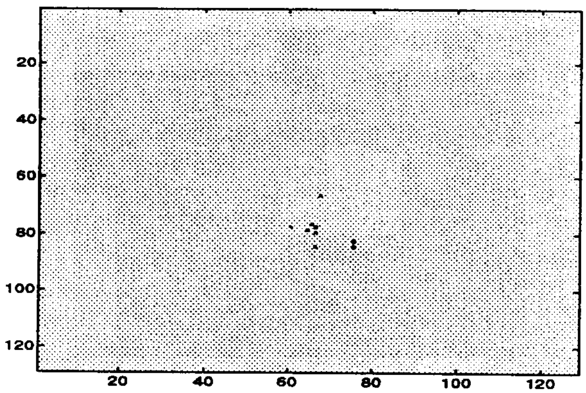

Figure 7: T-72 Scatters with RELAX

and 2-D MODE did not perform better than the DFT constrained method. For the images in Figures 1 and 3 these methods attained a local minimum or did not minimize 4. As these methods are known to work well for the case of sinusoids in white noise, we can only conclude that clutter and/or colored noise are adversely affecting their performance. In Figures 5, 6, and 7 of the scatter pixel locations we see that all methods locate the major scatter in the image. In the case of 2-D MODE the remaining poles model residual energy from the sidelobes of this scatter. In the case of 2-D IQML only distinct poles are selected in the second dimension which is a characteristic of the method.

For this data the RELAX method is more robust than 2-D IQML and 2-D MODE. The RELAX method starts with the DFT and then attains greater resolution by exploiting knowledge of the form of the signal. Subsequently, it performs better than the DFT bound and successfully locates the two major scatters in the image. As in the case of 2-D MODE the remaining poles model residual energy from the sidelobes of these scatters.

The results of this paper are based on image data for which the true signal content was not known. Thus the number of scatters that could be modeled as exponentials was unknown. We have used mean square error to identify how well the models fit the data. However, since the data may also contain clutter which is not well modeled by a small number of damped exponentials the lower performance bound of the SVD model may not be attainable with a purely exponential model. Therefore we hypothesize that a model which allows for colored noise may provide better results.

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