TRANSMULTIPLEXING OF MULTIDIMENSIONAL SIGNALS OVER ARBITRARY LATTICES WITH PERFECT RECONSTRUCTION *

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ABSTRACT

This research addresses the frequency multiplexing of multidimensional signals, having different bandwidths or defined on different lattices, with perfect or near perfect reconstruction i.e. zero or low crosstalk between signals and zero or low distortion of individual signals.

The paper discusses important issues such as: what are the valid modulating frequencies, how to manage quadrature modulation, what are the conditions for perfect reconstruction, how many signals can be transmitted perfectly, when and why linear periodically time-varying (LPTV) filters must be used in the system, structure of filters to preserve compatibility with conventional frequency multiplexers and design procedures.

1. INTRODUCTION

The application initially motivating this work is color television (NTSC and PAL) where the luminance component is frequency multiplexed with the quadrature modulated chrominance components. Existing systems have problems of crosstalk between the luminance and chrominance signals. The poor separation of signals leads to two annoying distortions: cross color (a rainbow pattern in parts of the image with high horizontal luminance, such as a referee's shirt) and cross luminance (the most serious problem with the comb filter receivers). Some recent methods [1] allow better management of the video spectrum by using multidimensional filters. However they still have a trade-off to make between the loss of resolution if low order filters are used (guard bands to keep) and ringing due to high order sharp filters (which are also expensive). Though not very objectionable on small TV sets, these artifacts become highly disturbing on large screen TV sets which have gained a lot of popularity in recent years. There is a definite need for an advanced NTSC system in which picture quality is improved while maintaining compatibility with all existing decoders.

Although the television problem is fairly specific, involving the multiplexing of three signals in two or three dimensions, our theory has been extended to a general theory of multi-D perfect reconstruction frequency multiplexing. Some parallels have been made between 1D transmultiplexers for M signals having equal bandwidths and subband coders [2]. But, there has been no complete study of multi-D transmultiplexers for signals defined on arbitrary lattices with the property of perfect reconstruction. Some practical considerations that are very important for multiplexers don't arise in subband coding [3] [4]. For instance, quadrature modulation which allows two signals to share the same frequency band would mean having two filters with the same passband in subband coding (usually they all have distinct passbands). Transmultiplexers must also consider modulating frequencies for signals; they must be carefully chosen for multi-D signals defined on arbitrary lattices. Conditions for compatibility with conventional multiplexers is also a concern that doesn't arise in subband coding. These practical issues, and others, motivate a serious study of transmultiplexers by themselves (and not just as dual problems of subband coding).

2. TRANSMULTIPLEXER DESCRIPTION

The proposed transmultiplexer system is shown in Fig. 1. The encoder starts with M signals critically sampled on their lattices Λ_i (in close relation with their bandwidth). $U_i(\mathbf{z}^{\Lambda_i})$ is the z-transform of $u_i(\mathbf{x})$ defined for $\mathbf{x} \in \Lambda_i$ as follows : $U_i(\mathbf{z}^{\Lambda_i}) = \sum_{\mathbf{x}\in\Lambda_i} u(\mathbf{x})\mathbf{z}^{-\mathbf{x}}$ with $\mathbf{z}^{\mathbf{x}} = \prod z_i^{x_i}$. This notation gives the information of the sampling lattice geometry which is not given in the conventional z-transform. Each signal is then pre-modulated such that its spectrum will be centered at the desired modulating frequency $\mathbf{f}_{c_i}$. This is done by multiplying each $u_i(\mathbf{x})$ by $cos(2\pi\mathbf{f}_{c_i}^T\mathbf{x})$. They are

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then upsampled to a common lattice Γ (for which each $\Lambda_i \subset \Gamma$) which acts as a transmission space lattice. The upsampling operation keeps the samples on Λ_i intact and inserts zeros for samples on $\Gamma \backslash \Lambda_i$. Signals are finally filtered by LPTV filters F_i and added together to form the composite transmitted signal $C(\mathbf{z}^\Gamma)$. Each F_i acts in two ways. First it is an interpolator because it generates samples in Γ from signal $S_{u_i}(\mathbf{z}^\Gamma) = S_i(\mathbf{z}^{\Lambda_i})$ which is really defined on Λ_i, but in a manner that the original signal will be recoverable even when mixed with others. Second, it acts as a passband filter by keeping only one replica (or two in case of quadrature modulation) in frequency of $S_{u_i}(\mathbf{z}^\Gamma)$. The LTI filters $F_{ij}(\mathbf{z}^\Gamma)$ making up the LPTV filter F_i in Fig. 2 must have approximately the same frequency response for frequency domain multiplexing behavior of the system. At reception, the decoder filters the composite signal through the LPTV filters H_i, which act as decimators (with about same frequency responses as their associated F_i), and downsamples to the original lattices Λ_i. Signals are then post-modulated to cancel the effect of pre-modulation and to finally obtain estimates of the initial signals.

The actual inputs and outputs of the transmultiplexer are the $S_i(\mathbf{z}^{\Lambda_i})$ and the $\hat{S}_i(\mathbf{z}^{\Lambda_i})$ respectively. Pre-modulation and post-modulation serve only to shift the spectra before transmission and after reception. The performance of the system depends on the sub-system from $S_i(\mathbf{z}^{\Lambda_i})$ to $\hat{S}_i(\mathbf{z}^{\Lambda_i})$.

3. CONDITIONS FOR PERFECT RECONSTRUCTION

The questions of interest now are : "When is it possible to get a perfect reconstruction system?" and "How can it be achieved?". To answer these questions, we consider the sub-system from $S_i(\mathbf{z}^{\Lambda_i})$ to $\hat{S}_i(\mathbf{z}^{\Lambda_i})$. A matrix representation of that sub-system can be derived. For it, all signals must be broken up into sub-signals over cosets on a common lattice, chosen as the intersection $\Lambda = \cap_{i=1}^{M} \Lambda_i$. Then for each coset, the use of distinct transmission filters and reception filters must be considered. This is like considering each coset of a signal as a different sub-signal to transmit and receive. Each signal "i" is broken up in $K_i = \frac{d(\Lambda)}{d(\Lambda_i)}$ such sub-signals. This explains the relatively complex structures of LPTV filters F_i and H_i shown in Fig. 2. Each filter $F_{ij}(\mathbf{z}^\Gamma)$ in that figure (because of its surrounding downsampler, upsampler and shifts $\mathbf{z}^{\mathbf{a}_{ij}}$ and $\mathbf{z}^{-\mathbf{a}_{ij}}$) assumes the transmission of a different coset (or sub-signal) $\Lambda + \mathbf{a}_{ij}$ of Λ_i. On the other hand, each filter $H_{ij}(\mathbf{z}^\Gamma)$ assumes the reception of a different coset $\Lambda + \mathbf{a}_{ij}$ of Λ_i. Define:

$$K_i = (\Lambda_i : \Lambda) = \frac{d(\Lambda)}{d(\Lambda_i)} \qquad (1)$$

$$\Lambda_i = \cup_{j=1}^{K_i} (\Lambda + \mathbf{a}_{ij}) \qquad (2)$$

$$N = (\Gamma : \Lambda) = \frac{d(\Lambda)}{d(\Gamma)} \qquad (3)$$

$$\Gamma = \cup_{j=1}^{N} (\Lambda + \mathbf{c}_j) \qquad (4)$$

$$F_{ij}(\mathbf{z}^\Gamma) = \sum_{l=1}^{N} \mathbf{z}^{\mathbf{c}_l + \mathbf{a}_{ij}} F_{ijl}(\mathbf{z}^\Lambda) \qquad (5)$$

$$H_{mn}(\mathbf{z}^\Gamma) = \sum_{k=1}^{N} \mathbf{z}^{-\mathbf{c}_k - \mathbf{a}_{mn}} H_{mnk}(\mathbf{z}^\Lambda) \qquad (6)$$

where $f_{ijl}(\mathbf{x}) = f_{ij}(\mathbf{x} - \mathbf{c}_l - \mathbf{a}_{ij})$, for $\mathbf{x} \in \Lambda$ and $h_{mnk}(\mathbf{x}) = h_{mn}(\mathbf{x} + \mathbf{c}_k + \mathbf{a}_{mn})$, $\mathbf{x} \in \Lambda$. We also define vector $\mathbf{S}(\mathbf{z}^\Lambda)$ and matrices $E(\mathbf{z}^\Lambda)$ and $R(\mathbf{z}^\Lambda)$ in the following manner:

$$[\mathbf{S}(\mathbf{z}^\Lambda)]_{\sum_{l=1}^{i-1} K_l + j} = S_{ij}(\mathbf{z}^\Lambda) \qquad (7)$$

$$[E(\mathbf{z}^\Lambda)]_{\sum_{l=1}^{m-1} K_l + n, k} = H_{mnk}(\mathbf{z}^\Lambda) \qquad (8)$$

$$[R(\mathbf{z}^\Lambda)]_{k, \sum_{l=1}^{i-1} K_l + j} = F_{ijk}(\mathbf{z}^\Lambda) \qquad (9)$$

From that an equivalent multi-input multi-output system, shown in Fig. 3, can be obtained. In that figure, $S_{ik}(\mathbf{z}^\Lambda)$ represents the k^{th} coset of signal i. Each column of matrix $R(\mathbf{z}^\Lambda)$ (row of $E(\mathbf{z}^\Lambda)$) contains the complete information (by using the given polyphase decompositions in (5) and (6) plus definitions in (8) and (9)) about a transmission (reception) filter $F_{ij}(\mathbf{z}^\Gamma)$ (or $H_{ij}(\mathbf{z}^\Gamma)$) corresponding to a given coset of a signal. A matrix representation of that system is as follows :

$$\hat{\mathbf{S}}(\mathbf{z}^\Lambda) = E(\mathbf{z}^\Lambda) R(\mathbf{z}^\Lambda) \mathbf{S}(\mathbf{z}^\Lambda) \qquad (10)$$

The matrix condition for perfect reconstruction is :

$$E(\mathbf{z}^\Lambda) R(\mathbf{z}^\Lambda) = I \qquad (11)$$

We usually are interested in FIR analysis/synthesis filter banks. We define the ring of FIR filters (causal and non-causal) over a lattice Λ as :

$$\mathcal{A}_p(\mathbf{z}^\Lambda) = \Big\{ A(\mathbf{z}^\Lambda) = \sum_{\mathbf{x} \in \Lambda} a(\mathbf{x}) \mathbf{z}^{\mathbf{x}} \mid a(\mathbf{x}) \in F, \; a(\mathbf{x}) = 0 \text{ except for finitely many terms} \Big\} \qquad (12)$$

where F is a field (which is usually chosen as the complex field for filters). The condition for perfect reconstruction implies that $E(\mathbf{z}^\Lambda)$ and $R(\mathbf{z}^\Lambda)$ must be unimodular. It follows that using LTI filters when a signal has many cosets would constrain columns of $R(\mathbf{z}^\Lambda)$ and rows of $E(\mathbf{z}^\Lambda)$ sometimes making perfect reconstruction impossible; this is why LPTV filters must be used. Since any product of elementary matrices is unimodular, solutions to (11) (infinitely many) are guaranteed to exist.

From the study of ranks of matrices $R(\mathbf{z}^\Lambda)$ and $E(\mathbf{z}^\Lambda)$, the dimensionality constraint on the number of signals that can be perfectly multiplexed is:

$$\sum_{i=1}^{M} d(\Lambda_i^*) \leq d(\Gamma^*) \qquad (13)$$

This condition is the same as for time multiplexing.

4. OBJECTIVE FILTERS AND PRACTICAL CONSIDERATIONS

In general we are interested in transmultiplexers that behave like a frequency multiplexer. In the application to NTSC, this is to maintain compatibility with conventional systems, i.e. existing TV sets should obtain acceptable pictures from a new transmultiplexer encoder while the new decoder would yield higher quality pictures. The transmultiplexer must thus possess some desired properties besides perfect reconstruction. For theory and design, we consider filters of the following form as objectives (i.e. very desirable but not an absolute must) for transmission and reception respectively:

$$f_{ij}(\mathbf{x}) = b_{ij}(\mathbf{x})cos[2\pi\mathbf{f}_{c_i}^T(\mathbf{x}-\frac{1}{2}\mathbf{d}_i)-\frac{m_i\pi}{2}] \qquad (14)$$

$$h_{kl}(\mathbf{x}) = p_{kl}(\mathbf{x})cos[2\pi\mathbf{f}_{c_k}^T(\mathbf{x}-\frac{1}{2}\mathbf{e}_k)-\frac{n_k\pi}{2}] \qquad (15)$$

where $b_{ij}(\mathbf{x})$ and $p_{kl}(\mathbf{x})$ are lowpass filters (usually having linear phase), $\frac{1}{2}\mathbf{d}_i$ and $\frac{1}{2}\mathbf{e}_k$ are their corresponding spatial (or spatiotemporal) centers, m_i and n_k are their modulation phases (integers). The $\mathbf{f}_{c_i}$ are the modulating frequencies of each signal. Considering perfect reconstruction from $S_i(\mathbf{z}^{\Lambda_i})$ to $\hat{S}_i(\mathbf{z}^{\Lambda_i})$, we have $\hat{U}_i(\mathbf{z}^{\Lambda_i}) = cos^2(2\pi\mathbf{f}_{c_i}^T\mathbf{x})U_i(\mathbf{z}^{\Lambda_i})$. For perfect reconstruction of the global system we need $cos^2(2\pi\mathbf{f}_{c_i}^T\mathbf{x}) = 1, \forall\mathbf{x} \in \Lambda_i$ implying that $2\mathbf{f}_{c_i} \in \Lambda_i^*$ (which we denote $\mathbf{f}_{c_i} \in \frac{1}{2}\Lambda_i^*$). Modulating frequencies are thus restricted.

Such filters ensure compatibility with existing systems since it can be proved that the system of Fig.1 with filters of the given forms behave like a conventional frequency multiplexer. When two signals possess the same sampling lattice, they are modulated in quadrature (i.e. they share the same band). Important theoretical results have been obtained for such signals (having $\Lambda_i = \Lambda_k$, $\mathbf{f}_{c_i} = \mathbf{f}_{c_k} \in \frac{1}{2}\Lambda_i^* \backslash \frac{1}{2}\Gamma^*$; note that a modulating frequency in $\frac{1}{2}\Gamma^*$ permits the presence of only one signal). For good recovery of signals, $n_i = -m_i$ and $\mathbf{e}_i = -\mathbf{d}_i$ must be chosen. These conditions arise whether or not quadrature modulation is used. For good separation of signals "i" and "k" modulated in quadrature, $m_i + n_k$ odd and $\mathbf{d}_i = -\mathbf{e}_k$ must be chosen.

This leads to necessary sets of conditions on filters to permit perfect reconstruction multiplexing. Furthermore, crosstalk between quadrature modulated signals is identical to zero if $\mathbf{d}_i \in 2\Gamma$ and $\mathbf{f}_{c_i} \in \frac{1}{4}\Gamma^* \backslash \frac{1}{2}\Gamma^*$ (which is what digital NTSC does).

5. DESIGN PROCEDURES

For the design of filter banks, we consider the minimization of a cost function depending on transmission and reception filters (of given orders) without any constraint. The optimization procedure is carried out in two steps. In the first one, we optimize filters for frequency response to be as close as possible as to the ones of the desired objective filters of equations (14) and (15) by using the procedure described by Lampropoulos and Fahmy [5]. They proved the problem to be convex and thus yielding a unique solution. In the second step, starting from the optimal frequency point (i.e. highly desirable filters), we add weight for perfect reconstruction error (which is not a convex problem since there are infinitely many solutions) to the previous cost function to find a near perfect reconstruction solution in the neighborhood of this starting point.

The software package to design transmultiplexers for 2-D and 3-D lattices has been written and tested. It will now be tested on 2-D and 3-D multiplexing schemes for NTSC video.

6. REFERENCES

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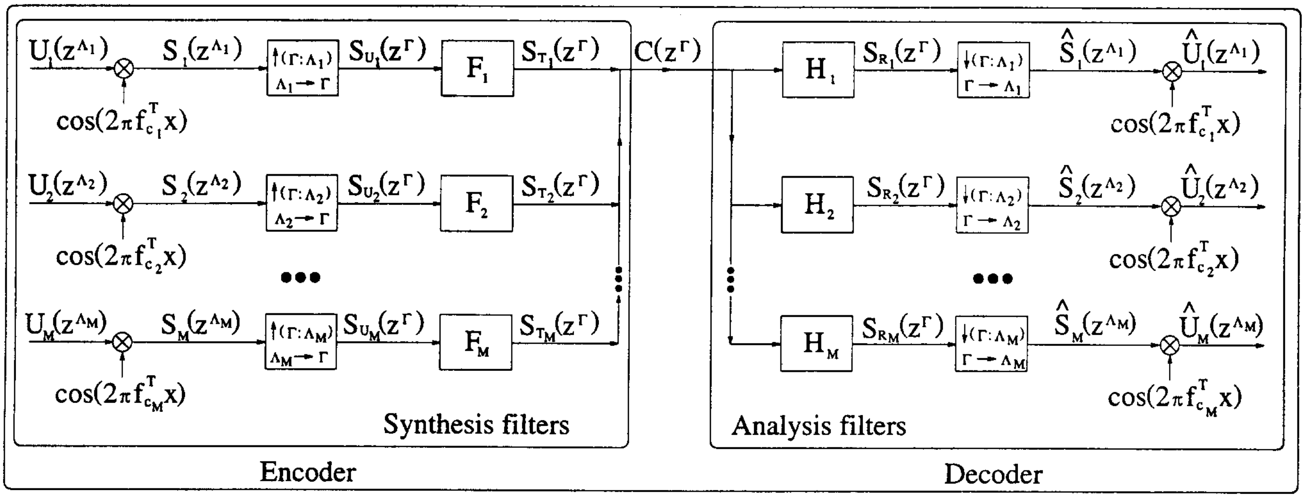

Figure 1: Multi-D Transmultiplexer system for M signals defined on arbitrary lattices

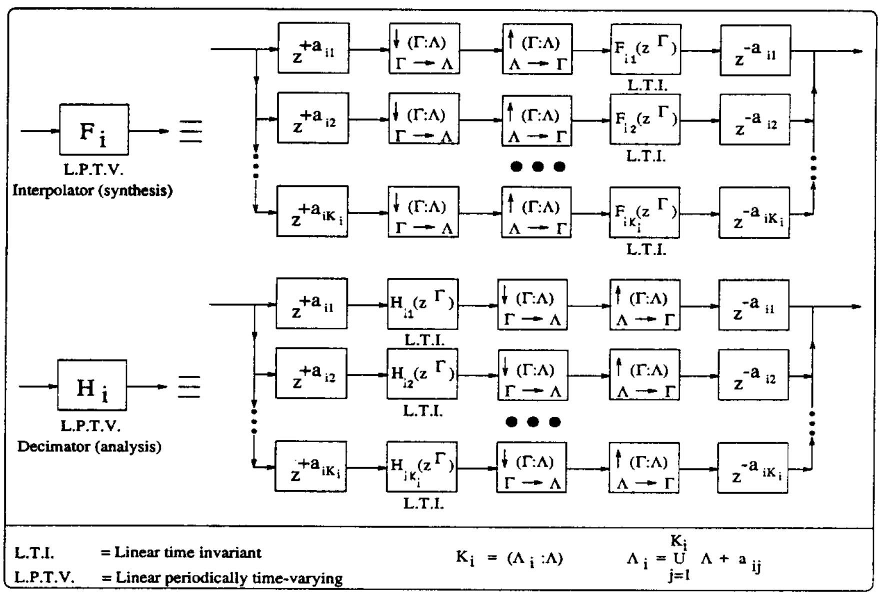

Figure 2: Structure of LPTV analysis/synthesis filters

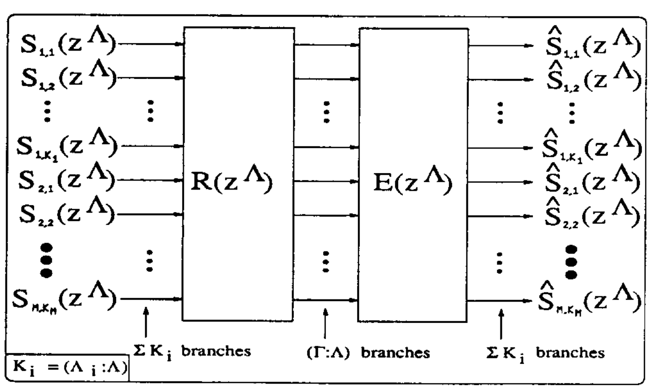

Figure 3: Multi-input multi-output system