Optimal Frequency Domain Design of Two-Dimensional Digital IIR Filters

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***Abstract*— A least-squares technique is presented for designing quarter-plane separable-denominator 2-D IIR filters to best approximate prescribed frequency domain (FD) specification. It is shown that the FD error vector is linearly related to the 2-D numerator coefficients whereas the relationship with the 2-D denominators is quasi-linear. Furthermore, the numerator and denominator estimation problems are theoretically decoupled. The quasi-linear relationship is used to formulate an algorithm for iterative estimation of the denominator. The numerator is found in one step using the estimated denominator. Computer simulations show the effectiveness of the proposed method and its superior performance compared to several existing methods.**

I. Introduction

Design of 2-D digital IIR filters from arbitrary frequency domain specifications is a highly nonlinear optimization problem [1-10], which can not be accomplished using classical techniques. Existing designs make use of variations of general nonlinear optimization methods, such as Newton-Raphson or Fletcher-Powell or linear programming to meet the prescribed design specifications [3, 4, 6-8]. But these general methods are computationally intensive, sensitive to the choice of initial estimates and may take large number of iterations. Also, none of these methods make use of the underlying matrix-structure inherent in the 2-D filter design problem. For Spatial Domain designs, it has been shown by several researchers (including the second author) that appropriate utilization of the underlying matrix structures leads to insightful theoretical framework and efficient computational algorithms [1, 2, 5, 9, 10]. In practice though, the filter specifications are usually in the frequency-domain and hence, direct design in the frequency domain is more desirable. The primary goal of this work is to demonstrate that an equivalent structured matrix framework does also exist in the frequency-domain which can be utilized equally effectively for designing 2D IIR filters.i We consider the design of denominator-separable filters here because the inherent symmetry in many commonly used 2D filters conforms to the separable-denominator structure and the stability of these filters can be easily verified.

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This work shows that the optimal 2D rational model identification problem belongs to a special class of *mixed*-nonlinear optimization problem where the linear and nonlinear parameters appear separately. Furthermore, the mixed nonlinear criterion can be decoupled into a purely linear problem for estimating the numerator and a separate nonlinear problem of reduced dimensionality, for estimating the separable denominators. The matrix structure of the nonlinear denominator criterion naturally leads to an iterative algorithm whereas the numerator is estimated with a single step of Least-Squares estimation. Extensive simulation studies show that the proposed approach produces superior frequency-domain match to prescribed magnitude response when compared with the results of various existing general approaches [6-8].

II. Problem Definition

The transfer function of a 2-D separable-denominator LSI system is given by

$$H(z_1, z_2) = \frac{\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a(i,j) z_1^{-i} z_2^{-j}}{\sum_{i=0}^{m_1} b(i) z_1^{-i} \sum_{j=0}^{m_2} c(j) z_2^{-j}}. \quad (1)$$

Let, $\mathbf{b} \triangleq [b(0)\ b(1) \cdots b(m_1)]^T$, $\mathbf{c} \triangleq [c(0)\ c(1) \cdots c(m_2)]^T$, $\mathbf{A} \triangleq \{a(i,j)\}$, for $i = 0, \ldots, n_1$ and $j = 0, \ldots, n_2$. Let the $k_1 \times k_2$ *desired* frequency response be

$$\mathbf{X}_d \triangleq \begin{bmatrix} x(\omega_{11}, \omega_{21}) & \cdots & x(\omega_{11}, \omega_{2k_2}) \\ \vdots & \ddots & \vdots \\ x(\omega_{1k_1}, \omega_{21}) & \cdots & x(\omega_{1k_1}, \omega_{2k_2}) \end{bmatrix} \quad (2)$$

and the frequency response of the filter at the same frequency points be $\mathbf{X}$. Let $\mathbf{x}_d \triangleq vec(\mathbf{X}_d)$ and $\mathbf{x} \triangleq vec(\mathbf{X})$. The problem considered in this paper is to estimate the coefficients in $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{A}$ by optimizing the following 2-D least-squares error criterion,

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$$\min_{\mathbf{b},\mathbf{c},\mathbf{A}} \|\mathbf{e}\|^2 \triangleq \|\mathbf{x}_d - \mathbf{x}\|^2, \text{ with } b(0) = c(0) = 1. \quad (3)$$

This criterion depends both on the 2-D denominator and numerator coefficients making the error minimization problem rather complicated. Utilizing the inherent matrix structures this problem is theoretically decoupled in the following section into two separate estimation problems of reduced dimensionalities.

III. Decoupling the error-criterion

Let $H_b(z_1)$ and $H_c(z_2)$ be the inverse filters of $B(z_1)$ and $C(z_2)$ respectively *i.e.*, $B(z_1)H_b(z_1) = 1$ and $C(z_2)H_c(z_2) = 1$. Then $H(z_1, z_2)$ can be rewritten as,

$$H(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1)\,C(z_2)} = H_b(z_1)\,A(z_1, z_2)\,H_c(z_2) \quad (4)$$

For $k_1 \times k_2$ significant spatial samples, the above relation can be expressed in matrix notation as

$$\mathbf{H} = \mathbf{H}_L^b \mathbf{A} \mathbf{H}_L^{cT} \quad (5)$$

where, $\mathbf{H} \triangleq \{h(i,j)\} \in \mathbb{R}^{k_1 \times k_2}$,

$$\mathbf{H}_L^b(i,j) \triangleq \begin{cases} h_b(i-j), & \text{for } (i-j) \geq 0 \\ 0, & \text{for } (i-j) < 0 \end{cases} \quad (6)$$

for $i = 1, \ldots, k_1$, $j = 1, \ldots, (n_1 + 1)$ and

$$\mathbf{H}_L^c(i,j) \triangleq \begin{cases} h_c(i-j), & \text{for } (i-j) \geq 0 \\ 0, & \text{for } (i-j) < 0 \end{cases} \quad (7)$$

for $i = 1, \ldots, k_2$, $j = 1, \ldots, (n_2 + 1)$. The frequency response of the 2-D filter can be written in a matrix-decomposed form as,

$$\mathbf{X} = \mathbf{W}_b \mathbf{H} \mathbf{W}_c^T, \qquad \text{where}, \quad (8)$$

$$\mathbf{W}_b \triangleq \begin{bmatrix} 1 & e^{-j\omega_{11}} & \ldots & e^{-j(k_1-1)\omega_{11}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\omega_{1k_1}} & \ldots & e^{-j(k_1-1)\omega_{1k_1}} \end{bmatrix}, \quad (9)$$

$$\mathbf{W}_c \triangleq \begin{bmatrix} 1 & e^{-j\omega_{21}} & \ldots & e^{-j(k_2-1)\omega_{21}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\omega_{2k_2}} & \ldots & e^{-j(k_2-1)\omega_{2k_2}} \end{bmatrix}. \quad (10)$$

Applying the *vec* operator on both sides of (8) and using Kronecker product ($\otimes$) representation, we get

$$\begin{aligned} vec(\mathbf{X}) \triangleq \mathbf{x} &= vec(\mathbf{W}_b \mathbf{H} \mathbf{W}_c^T) = vec(\mathbf{W}_b \mathbf{H}_L^b \mathbf{A} \mathbf{H}_L^{cT} \mathbf{W}_c^T) \\ &= (\mathbf{W}_c \mathbf{H}_L^c \otimes \mathbf{W}_b \mathbf{H}_L^b) vec(\mathbf{A}) \quad (11) \\ &= (\mathbf{W}_c \mathbf{H}_L^c \otimes \mathbf{W}_b \mathbf{H}_L^b)\mathbf{a}. \quad (12) \end{aligned}$$

Hence, the error between the desired and the filter frequency response, as defined in (3), can be written as,

$$\mathbf{e} = \mathbf{x}_d - \mathbf{x} = \mathbf{x}_d - (\mathbf{W}_c \mathbf{H}_L^c \otimes \mathbf{W}_b \mathbf{H}_L^b)\mathbf{a}. \quad (13)$$

This expression shows explicitly that the frequency domain error is linearly related to the numerator vector $\mathbf{a}$ and nonlinearly related to the denominators in a rather complicated manner. Interestingly, if the denominator coefficients are known, the least-squares estimate of the numerator coefficients can be obtained by minimizing the error in (3) which produces,

$$\mathbf{a} = (\mathbf{W}_c \mathbf{H}_L^c \otimes \mathbf{W}_b \mathbf{H}_L^b)^{\#} \mathbf{x}_d \quad (14)$$

where # denotes the pseudo-inverse.Substituting this in (12), we get the *decoupled* denominator criterion,

$$\|\mathbf{e}(\mathbf{b},\mathbf{c})\|^2 = \|(\mathbf{I}_{k_1k_2} - (\mathbf{P}_{\mathbf{W}_c\mathbf{H}_L^c} \otimes \mathbf{P}_{\mathbf{W}_b\mathbf{H}_L^b}))\mathbf{x}_d\|^2 \quad (15)$$

where, $\mathbf{P}_\mathbf{Y} \triangleq \mathbf{Y}(\mathbf{Y}^H\mathbf{Y})^{-1}\mathbf{Y}^H$ denotes the projection matrix of a matrix $\mathbf{Y}$ with H being the conjugate-transpose operator. Extending Theorem 2.1 in [12], it can be shown that if the denominator is estimated by minimizing the criterion in (15) and that estimate is used in (14), the estimates retain the global optima of the original criterion in (3).

IV. Reparametrization of the error-criterion

In this section the *decoupled* criterion in (15) will be directly related to the denominator coefficients. The inverse filter relation $B(z_1)H_b(z_1) = 1$ can be expressed in matrix notation as

$$\mathbf{B}_L \mathbf{H}_b = \mathbf{I}_{k_1}, \qquad \text{where}, \quad (16)$$

$$\mathbf{B}_L(i,j) \triangleq \begin{cases} b(i-j), & \text{if } (i-j) \geq 0 \\ 0, & \text{if } (i-j) < 0 \end{cases} \triangleq \begin{bmatrix} \mathbf{B}_u \\ --- \\ \mathbf{B}^T \end{bmatrix}, \quad \text{and} \quad (17)$$

$$\mathbf{H}_b(i,j) \triangleq \begin{cases} h_b(i-j), & \text{if } (i-j) \geq 0 \\ 0, & \text{if } (i-j) < 0 \end{cases} \triangleq [\mathbf{H}_L^b \mid \mathbf{H}_R^b], \quad (18)$$

for $i = 1, \ldots, k_1$ and $j = 1, \ldots, k_1$, with the partitions being such that $\mathbf{B}_u \in \mathbb{R}^{(n+1)\times k_1}$ and $\mathbf{H}_L^b \in \mathbb{R}^{k_1 \times (n+1)}$. Let, $\mathbf{W}_{b_I}\mathbf{W}_b = \mathbf{I}_{k_1}$. This inverse exists because the frequency points ω_{lm}'s are distinct. Using this inverse in (16) along with the partitions in (17) and (18),

$$\begin{aligned} \mathbf{B}_L \mathbf{W}_{b_I} \mathbf{W}_b \mathbf{H}_b &= \mathbf{I}_{k_1} \\ &= \begin{bmatrix} \mathbf{B}_u \\ -- \\ \mathbf{B}^T \end{bmatrix} \mathbf{W}_{b_I} \mathbf{W}_b [\mathbf{H}_L^b \mid \mathbf{H}_R^b] \quad (19) \end{aligned}$$

$$= \begin{bmatrix} \mathbf{B}_u \mathbf{W}_{b_I} \mathbf{W}_b \mathbf{H}_L^b & | & \mathbf{B}_u \mathbf{W}_{b_I} \mathbf{W}_b \mathbf{H}_R^b \\ ------- & | & ------- \\ \mathbf{B}^T \mathbf{W}_{b_I} \mathbf{W}_b \mathbf{H}_L^b & | & \mathbf{B}^T \mathbf{W}_{b_I} \mathbf{W}_b \mathbf{H}_R^b \end{bmatrix} \qquad (20)$$

The bottom-left corner element of the matrix at right suggests that $\mathbf{W}_{b_I}^H \mathbf{B}$ and $\mathbf{W}\mathbf{H}_L^b$ are orthogonal, *i.e.*, $(\mathbf{W}_{b_I}^H \mathbf{B})^H (\mathbf{W}\mathbf{H}_L^b) = \mathbf{0}$. Also, since $rank(\mathbf{W}_{b_I}^H \mathbf{B}) + rank(\mathbf{W}\mathbf{H}_L^b) = (k_1 - n_1 - 1) + (n_1 + 1) = k_1$, using a theorem on projection matrices,

$$\mathbf{P}_{\mathbf{W}_{b_I}^H \mathbf{B}} + \mathbf{P}_{\mathbf{W}\mathbf{H}_L^b} = \mathbf{I}_{k_1}. \qquad (21)$$

Similarly, from the inverse filter relation $C(z_2)H_c(z_2) = 1$, we can get

$$\mathbf{P}_{\mathbf{W}_{c_I}^H \mathbf{C}} + \mathbf{P}_{\mathbf{W}\mathbf{H}_L^c} = \mathbf{I}_{k_2}. \qquad (22)$$

Substituting the above relations in (15), the error can be written (after some algebraic manipulations) as

$$\begin{aligned} \mathbf{e}_d &\triangleq [(\mathbf{I}_{k_2} - \mathbf{P}_{\mathbf{W}_{c_I}^H \mathbf{C}}) \otimes \mathbf{P}_{\mathbf{W}_{b_I}^H \mathbf{B}} + \mathbf{P}_{\mathbf{W}_{b_I}^H \mathbf{B}} \otimes \mathbf{I}_{k_1}]\mathbf{x}_d \\ &= [((\mathbf{I}_{k_2} - \mathbf{P}_{\mathbf{W}_{c_I}^H \mathbf{C}}) \otimes \mathbf{V}_b)\mathbf{X}^1 \ \ (\mathbf{V}_c \otimes \mathbf{I}_{k_1})\mathbf{X}^2] \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix}. \end{aligned} \qquad (23)$$

where, $\mathbf{X}^1$ and $\mathbf{X}^2$ are formed with prescribed data [13],

$$\mathbf{V}_b \triangleq (\mathbf{W}_{b_I}^H \mathbf{B})((\mathbf{W}_{b_I}^H \mathbf{B})^H (\mathbf{W}_{b_I}^H \mathbf{B}))^{-1} \text{ and} \qquad (24)$$

$$\mathbf{V}_c \triangleq (\mathbf{W}_{c_I}^H \mathbf{C})((\mathbf{W}_{c_I}^H \mathbf{C})^H (\mathbf{W}_{c_I}^H \mathbf{C}))^{-1}. \qquad (25)$$

The final equation in (23) clearly shows the *quasi*-linear relationship that the decoupled error vector $\mathbf{e}_d$ has with the denominator vector $\begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix}$. Similar to the work in [9-11], this relationship can be utilized to estimate the denominators iteratively. Details are omitted due to lack of space and will be reported in [13].

V. Simulation Results

Several designs were implemented using the proposed algorithm and the performances were compared with existing approaches. Fig. 1 and 2 show the results of two Bandpass filter examples from [6, 7] and Fig. 3 shows a Lowpass case [8]. Fig. 1a, 2a and 3a show results using the methods proposed in [6], [7] and [8], respectively. For the same or less numerator/denominator orders, Fig. 1b, 2b and 3b show the corresponding results using the proposed method. The relative rms errors [7] for the results in Fig. 1a, 2a and 3a are 0.67, 0.28 and 0.77, respectively. The errors for the results in Fig. 1b, 2b and 3b are 0.21, 0.26 and 0.68 respectively. Clearly, the proposed approach found better match with lesser number of coefficients, in all cases. The number of iterations for the proposed approach was less than 10 in all cases, whereas the general optimization approaches sometimes took close to hundred or more iterations.

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Other Methods

Proposed Method

Fig. 1a : Method in [6]

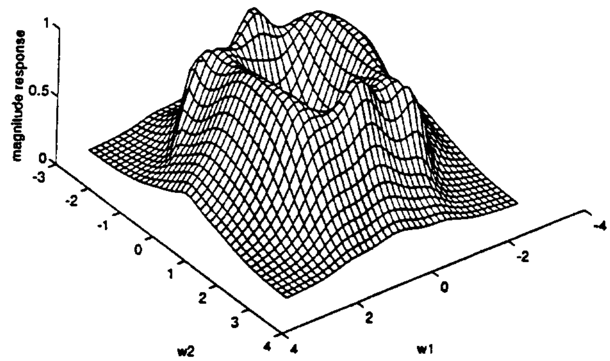

Fig. 1b

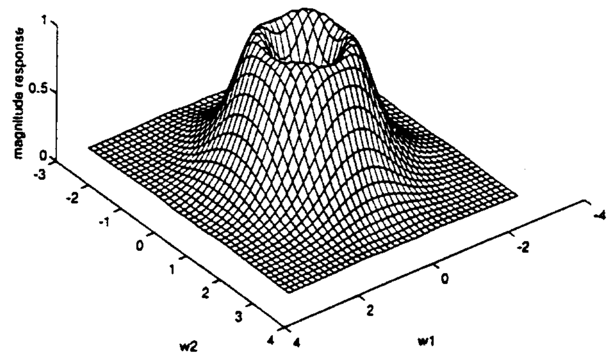

Fig. 2a : Method in [7]

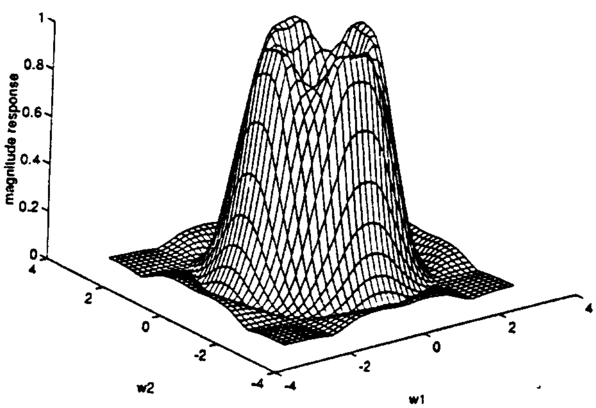

Fig. 2b

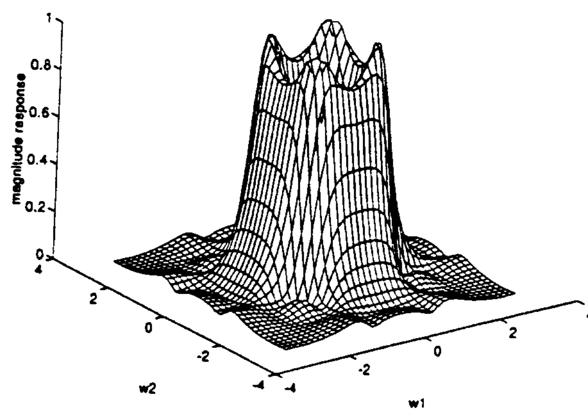

Fig. 3a : Method in [8]

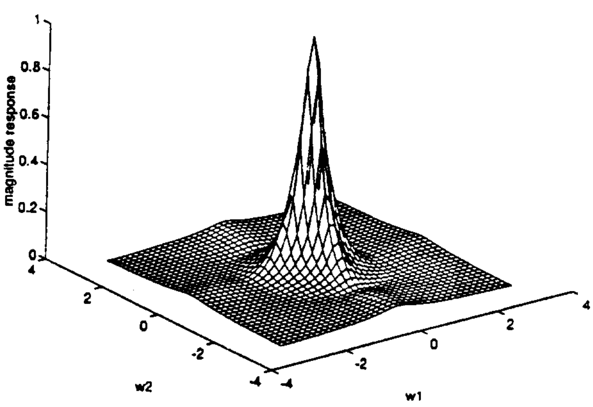

Fig. 3b