Localization of Multiple Signals Using Subarrays Data

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Abstract

A new technique for localization of multiple signals is presented. Unlike existing techniques which require that the whole array be sampled simultaneously and consequently require many receivers, our technique allows to sample arbitrary subarrays sequentially and consequently significantly reduces the required number of receivers. The estimation method we use in conjuction with this sampling scheme is based on approximating the corresponding maximum likelihood estimator by a computationally simpler Generalized Least Squares(GLS) estimator that is proved to be both consistent and efficient.

1 Introduction

We consider the problem of localizing multiple narrowband signals by an arbitrary sensor array. The existing super-resolution solutions to this problem require that the whole array be sampled simultaneously and consequently require that the number of receivers equal the number of sensors. This requirement may impede practical implementation of such solutions especially in cases where the number of sensors is large and where the receivers are expensive. We propose in this paper a different approach where the array is sampled by parts, i.e. different subarrays are sampled sequentially, with the subarrays being arbitrary and not necessarily mutually exclusive. Using this approach, and keeping the size of the subarrays small, the number of receivers needed, and consequently the associated hardware involved, is significantly smaller than in the common approach. Thus, for instance, only two receivers may suffice to sample the output of a large array if different pairs of sensors are sequentially switched to their inputs. In fact, the general idea is not new: conventional interferometers are using such a switching technique to sample different "baselines" of the array.

To illuminate the difficulties involved in performing detection and localization of multiple sources from subarrays data it is instructive to consider the case where pairs of sensors are sampled sequentially, i.e. each subarray consists of two sensors only. Notice first that in this case no single subarray can perform detection or localization by its own. Moreover, when only part of the different pairs of sensors are sampled, the existing high-resolution techniques are not applicable since no estimate of the whole array covariance, nor of any large subarray, can be constructed. Even if all the different pairs of sensors are sampled, and an estimate of the whole array covariance matrix $\hat{\mathbf{R}}$ is obtained in an element-by-element manner, with the (i,j)-th element's estimate, $\hat{R}_{ij}$, obtained from the samples of the (i,j)-th sensor pair, eignvalue-based detection criteria, such as [6], may fail since the eignvalues of $\hat{\mathbf{R}}$ are not guaranteed to be non-negative (as is the case when the whole array is sampled simultaneously).

The estimation method we use in conjuction with the subarrays sampling scheme is based on first deriving the Maximum Likelihood Estimator (MLE) for the problem, and then, since the computational load involved in its implementation is too heavy, approximating it by a Generalized Least Squares(GLS) estimator that is computationally much simpler. This new estimator is proved to be both consistent and efficient. The estimator is applicable also to the important case of coherent signals arising, for instance, in specular multipath propagation.

2 Problem Formulation

Consider q wave-fronts impinging from locations $\theta_1, \ldots, \theta_q$ on an array consisting of p sensors. For simplicity assume that the sensors and the sources are all located on the same plane and that the sources are in the far-field of the array, so that $\{\theta_i\}$ represent the Directions-Of-Arrival (DOAs).

Assume also that the sources emit narrow-band signals all centered around a common frequency. Let $s_i(t)$ denote the complex envelope of the i-th source signal, and let $\mathbf{x}_c(t) = (x_{c1}(t), x_{c2}(t), \ldots, x_{cp_c}(t))^T$ denote the vector of complex envelopes formed from the signals received by the c-th subarray of sensors, with p_c denoting

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the number of sensors in the subarray, and T denoting transposition. In the presence of additive noise, this received vector can be expressed as:

$$\mathbf{x}_c(t) = \sum_{k=1}^{q} \mathbf{a}_c(\theta_k)s_k(t) + \mathbf{n}_c(t) \tag{1}$$

where $\mathbf{a}_c(\theta)$ is the steering vector of the subarray expressing its complex response to a planar wavefront arriving from direction θ, and $\mathbf{n}_c(t)$ is the complex envelope of the c-th subarray noise. This expression can be written more compactly as:

$$\mathbf{x}_c(t) = \mathbf{A}_c(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{n}_c(t) \tag{2}$$

where $\boldsymbol{\theta} \stackrel{\text{def}}{=} (\theta_1 \dots \theta_q)^T$, and $\mathbf{A}_c(\boldsymbol{\theta}) \stackrel{\text{def}}{=} [\mathbf{a}_c(\theta_1), \dots, \mathbf{a}_c(\theta_q)]$ is a $p_c \times q$ matrix, and $\mathbf{s}(t) \stackrel{\text{def}}{=} (s_1(t), \dots, s_q(t))^T$ is a vector formed from the emitted signals. We shall assume that the steering vectors $\{\mathbf{a}_c(\theta)\}$ are known for all c and all $\theta \in \boldsymbol{\Theta}$, where $\boldsymbol{\Theta}$ denotes the field-of-view.

Let the subarrays be sampled sequentially, with $\mathbf{X}_c \stackrel{\text{def}}{=} [\mathbf{x}_c(t_1^c), \dots, \mathbf{x}_c(t_{m_c}^c)]$ denoting the c-th subarray samples, m_c denoting the number of samples taken from this subarray, and $t_1^c, \dots, t_{m_c}^c$ denoting the sampling instants.

Now, the problem is stated as follows: Given the samples of K different subarrays $\mathbf{X} \stackrel{\text{def}}{=} \{\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_K\}$ - estimate the directions $\boldsymbol{\theta}$.

To solve this problem, we make the following assumptions:

*A*1. The noise-vector $\mathbf{n}(t)$ of the whole array is a zero-mean complex-Gaussian wide-sense stationary process with a covariance matrix $\sigma^2\mathbf{I}$, where σ^2 is an unknown positive scalar and $\mathbf{I}$ is the identity matrix, and $\{\mathbf{n}(t_i^c)\}$ are uncorrelated $\forall c, i$.

*A*2. The signal-vector $\mathbf{s}(t)$ is a zero-mean complex-Gaussian wide-sense stationary process uncorrelated with the noise-vector and having an unknown arbitrary covariance matrix $\mathbf{P}$. The signal samples $\{\mathbf{s}(t_i^c)\}$ are uncorrelated $\forall c, i$.

Based on these assumptions and using (2), the covariance matrix of the c-th subarray is given by

$$\mathbf{R}_c(\phi) = \mathbf{A}_c(\theta)\mathbf{P}\mathbf{A}_c^H(\theta) + \sigma^2\mathbf{I} \tag{3}$$

where ϕ represents all the unknown parameters $\phi = \{\boldsymbol{\theta}, \mathbf{P}, \sigma^2\}$, and $()^H$ denotes complex-conjugate transposition.

The pdf of the whole batch of data $\mathbf{X}$ is the product of the subarray pdfs, that is

$$p(\mathbf{X} \mid \boldsymbol{\theta}, \mathbf{P}, \sigma^2) = \prod_{c=1}^{K} \{\pi^{-p_c}|\mathbf{R}_c|^{-1} \exp[-\mathrm{tr}(\mathbf{R}_c^{-1}\hat{\mathbf{R}}_c)]\}^{m_c} \tag{4}$$

where $|\cdot|$ denotes determinant, tr() denotes trace, $\hat{\mathbf{R}}_c \stackrel{\text{def}}{=} \mathbf{X}_c\mathbf{X}_c^H/m_c$ is a sample-covariance matrix. The MLE is given by:

$$\text{MLE:} \qquad \hat{\boldsymbol{\theta}} = \arg \max_{\boldsymbol{\theta}, \mathbf{P}, \sigma^2} [p(\mathbf{X} \mid \boldsymbol{\theta}, \mathbf{P}, \sigma^2)] \tag{5}$$

Unfortunately, due to the large number of free parameters involved, the computational load involved in this maximization is very heavy. Yet, unlike in the MLE estimator for simultaneous sampling [2], we were not able, except for the case of a single source, to reduce the computational load by eliminating $\mathbf{P}$ and σ^2 analytically. For the case of a single source, the exact reduction of (5) to a one-dimensional maximization problem is presented in [5].

3 The GLS Estimator

The estimator we propose minimizes the "distance" between the measured sample-covariances, $\{\hat{\mathbf{R}}_c\}$, and the modelled covariances $\{\mathbf{R}_c\}$ given in (3):

$$\hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}, \mathbf{P}, \sigma^2} \sum_{c=1}^{K} \|\mathbf{T}_c[\hat{\mathbf{R}}_c - \mathbf{A}_c(\boldsymbol{\theta})\mathbf{P}\mathbf{A}_c^H(\boldsymbol{\theta}) - \sigma^2\mathbf{I}]\mathbf{T}_c^H\|_F^2 \tag{6}$$

where $\|\cdot\|_F$ denotes the Frobenius norm, and $\mathbf{T}_c$ denotes a "whitening" transformation that "whitens" the elements of the observed error matrix $(\hat{\mathbf{R}}_c - \mathbf{R}_c)$. This transformation can be approximated [5] by $\mathbf{T}_c = \sqrt[4]{m_c}\,\hat{\mathbf{R}}_c^{-1/2}$. The above estimator can be regarded as a variant of what in the statistical literature [3] [1] is known as the Generalized Least Squares(GLS) estimator.

To solve the above minimization problem, we first minimize with respect to $\mathbf{P}$ and σ^2 while holding $\boldsymbol{\theta}$ fixed. Notice that since (6) is quadratic with respect to their elements, this minimization can be carried out analytically. We then substitute the minimizing values $\hat{\mathbf{P}}(\boldsymbol{\theta})$ and $\hat{\sigma}^2(\boldsymbol{\theta})$ back into the cost function (6) and obtain a cost function that is a function of $\boldsymbol{\theta}$ only. The result is [5]:

$$\text{GLS:} \qquad \hat{\boldsymbol{\theta}} = \arg \min_{\boldsymbol{\theta}} \left\{\mathbf{r}^T \mathbf{P}^{\perp}_{\mathcal{A}(\boldsymbol{\theta})}\mathbf{r}\right\} \tag{7}$$

where

$$\mathbf{r} \stackrel{\text{def}}{=} \begin{bmatrix} \mathrm{vec}(\mathbf{I}_1) \\ \vdots \\ \mathrm{vec}(\mathbf{I}_K) \end{bmatrix}$$

$$\mathcal{A}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \begin{bmatrix} \tilde{\mathbf{A}}_1^*(\boldsymbol{\theta}) \otimes \tilde{\mathbf{A}}_1(\boldsymbol{\theta}) & \sqrt{m_1}\,\mathrm{vec}(\hat{\mathbf{R}}_1^{-1}) \\ \vdots & \vdots \\ \tilde{\mathbf{A}}_K^*(\boldsymbol{\theta}) \otimes \tilde{\mathbf{A}}_K(\boldsymbol{\theta}) & \sqrt{m_K}\,\mathrm{vec}(\hat{\mathbf{R}}_K^{-1}) \end{bmatrix}$$

with $\mathbf{I}_c$ denoting the $p_c \times p_c$ identity matrix, $*$ denoting the complex conjugate, $\otimes$ denoting the Kronecker product, $\tilde{\mathbf{A}}_c(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \sqrt[4]{m_c}\, \hat{\mathbf{R}}_c^{-1/2} \mathbf{A}_c(\boldsymbol{\theta})$, and where $\mathbf{P}^{\perp}_{\mathcal{A}(\boldsymbol{\theta})}$ is the projection matrix on the subspace orthogonal to the subspace spanned by the columns of $\mathcal{A}(\boldsymbol{\theta})$: $\mathbf{P}^{\perp}_{\mathcal{A}(\boldsymbol{\theta})} \stackrel{\text{def}}{=} \mathbf{I} - \mathcal{A}(\boldsymbol{\theta})(\mathcal{A}^{\mathbf{H}}(\boldsymbol{\theta})\mathcal{A}(\boldsymbol{\theta}))^{-1}\mathcal{A}^{\mathbf{H}}(\boldsymbol{\theta})$.

Notice that the dimensionality of the minimization problem has been reduced to q. The minimization can be efficiently accomplished by any multidimensional minimization technique.

The above estimator can be proved [5] to be both consistent, and asymptotically efficient (i.e. it asymptotically attains the Cramer-Rao lower bound (CRB)).

In the above GLS estimator, $\hat{\mathbf{P}}$ has not been constrained to be Hermitian, and neither has $\hat{\sigma}^2$ been constrained to be real-valued. This could however be easily accomplished [5] by inserting into (6) $\mathbf{P} = \mathbf{P}_R + j\mathbf{P}_I$, where $\mathbf{P}_R$ and $\mathbf{P}_I$ denote the real and imaginary parts of $\mathbf{P}$, and constraining $\mathbf{P}_R$ and $\mathbf{P}_I$ to be symmetric and anti-symmetric, respectively, thus reducing the total number of unknown components in $\mathbf{P}$ by half. The resulting estimator is more accurate and computationally more effective due to the reduction of the parameters dimensionality.

In case the signals are known to be uncorrelated, $\mathbf{P}$ can be taken to be diagonal thus reducing the number of unknown components, and consequently improving the estimator's performance [5].

4 Simulation results

To demonstrate the performance of the described method we simulated a 5-omnidirectional-element uniform circular array with a diameter of 0.6λ. The 3-db beamwidth of the array is approximately 72°. Two equipower uncorrelated Gaussian sources were located at 100° and 120°. Prior knowledge about the uncorrelatedness of the sources is assumed to be available. The subarrays consisted of the five longest 2-element subarrays (baselines). 128 independent samples were taken from each subarray. In addition to the GLS estimator described above, we also examined an estimator obtained by substituting the GLS estimates $\hat{\mathbf{P}}(\boldsymbol{\theta})$ and $\hat{\sigma}^2(\boldsymbol{\theta})$ into the likelihood function, thus obtaining a concentrated likelihood that is a function of $\boldsymbol{\theta}$ only. We call the minimizer of this concentrated likelihood the GLS-ML estimator.

In a first experiment A set of 100 monte-carlo runs was carried out for each SNR, with the DOAs estimated in each run and the RMS DOA error (averaged over the two sources) computed from the whole set. Fig.1 compares the results obtained by using the GLS and GLS-ML estimators with the CRB (derived in [5]). The efficiency of the estimators is evident.

In a second experiment we sampled all the ten possible 2-element subarrays (baselines), but the number of samples from each subarray was reduced by a factor of two, i.e. $m_c = 64$, thus retaining the same overall number of samples used. The results are shown in Fig.2. Comparing Fig.2 with Fig.1, it turns out, not surprisingly, that it is advantageous to sample only the largest baselines rather than spend time on the smaller baselines.

Interestingly enough, notice that in Figs.1 and 2 the DOA errors do not vanish for high SNR values. This phenomenon is seen more clearly in Figs.3 and 4 where the CRB for subarray sampling is compared with the CRB for simultaneous sampling. The number of snapshots in the simultaneous sampling is $m = 64$, i.e. equal to the number of samples taken from each subarray. Notice that at low SNR there is essentially no difference in the DOA errors of the two DF methods.

To demonstrate the advantage of the proposed estimators we compared them with a "naive" estimator constructed as follows: from the various 2-elements samples we create a "sample-covariance" $\hat{\hat{\mathbf{R}}} = \{\hat{\hat{R}}_{ij}\}$ of the whole array in an element-by-element way, with $\hat{\hat{R}}_{ij} = \hat{\hat{R}}^*_{ji}$ created from the samples of the subarray containing elements i and j, and $\hat{\hat{R}}_{ii}$ created from the samples of all the subarrays containing element i. We then use the Alternating Projections algorithm [7] in conjuction with $\hat{\hat{\mathbf{R}}}$ to estimate the DOAs. (In fact any other estimation method, such as MUSIC [4], could have been used instead). It turned out that the RMS error exceeded 40° at all SNR values. That is, the sources are not resolvable by this naive estimator.

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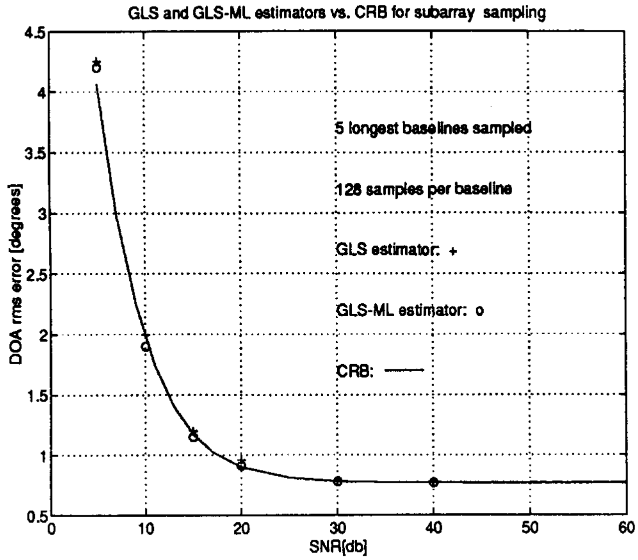

Figure 1: The RMS DOA error of subarray sampling as a function of SNR for two equipower uncorrelated sources located at 100° and 120° impinging on a 5-element circular array with diameter 0.6λ. Only the 5 longest baselines are sampled.

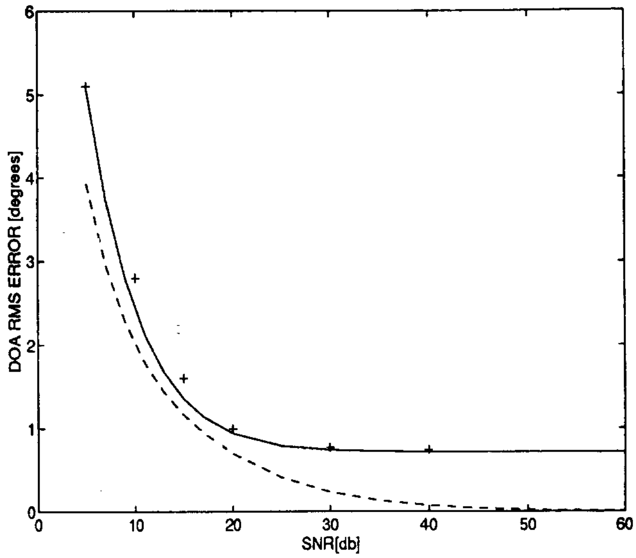

Figure 3: The RMS DOA error as a function of SNR for two equipower uncorrelated sources located at 100° and 120° impinging on a 5-element circular array with diameter 0.6λ. The solid line represents the CRB for subarray sampling of all the 10 baselines. The dashed line represents the CRB for simultaneous sampling of the whole array. The + signs represent the results obtained with the GLS estimator.

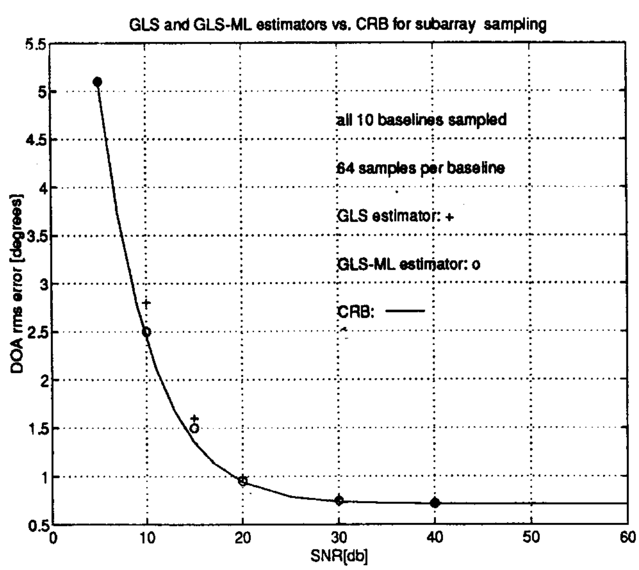

Figure 2: The RMS DOA error as a function of SNR for two equipower uncorrelated sources located at 100° and 120° impinging on a 5-element circular array with diameter 0.6λ. All the 10 baselines are consecutively sampled.

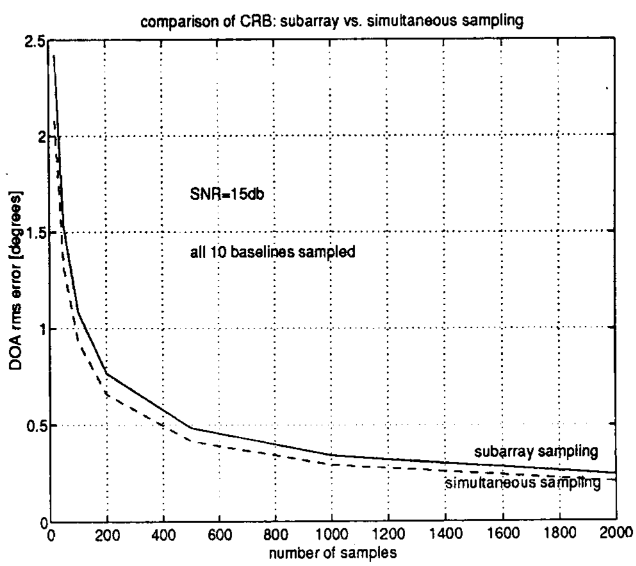

Figure 4: The RMS DOA error as a function of the number of samples for two equipower uncorrelated sources located at 100° and 120° impinging on a 5-element circular array with diameter 0.6λ. The solid line represents the CRB for subarray sampling of all the 10 baselines. The dashed line represents the CRB for simultaneous sampling of the whole array.