Localization of Correlated and Uncorrelated Signals in Colored Noise via Generalized Least Squares

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Abstract

A new method for localizing multiple signals in spatially-colored background noise using an arbitrary passive sensor array is presented. The method enables also to exploit prior knowledge that the signals are uncorrelated, in case such information is available, so as to improve the performance and allow localization even if the number of signals exceeds the number of sensors. The estimation is based on the Generalized Least Squares criterion, and is both consistent and efficient. Simulation results confirming the theoretical results are included.

1 Introduction

We consider the problem of localizing multiple signals in unknown spatially-colored background noise using an arbitrary passive sensor array. Only relatively few techniques have been developed to handle colored noise. In [5], an approach was presented, based on an assumption that the noise field is invariant to array displacement. In [15] a MAP approach assuming a completely unknown Hermitian positive definite noise covariance matrix was introduced. This estimator was shown in [7] to be inconsistent, except in special cases. In [13] an MDL approach was proposed, but it also may yield inconsistent estimates. In [10] and [4] methods suitable to linear uniform arrays are proposed, based on an AR model for the noise. A spatial ARMA model was employed in [3], and the noise covariance estimated as a preleminary step. In [9] an instrumental variable approach is developed, restricted to the case where the correlation time of the signals is longer than that of the noise. The method described in [14] and [2] is based on finding the parameters that give the "best fit" between a modelled covariance and the sample covariance.

In this paper we present a new technique based on modeling the noise covariance by a a linear parametric model. This technique is close in spirit to the method described in [14] and [2], however, we use a different criterion for the goodness-of-fit, and consequently get an estimator that is proved to be asymptotically efficient. Also, our estimator is computationally simpler.

An additional and different subject investigated in this paper is the exploitation of prior knowledge that the signals are uncorrelated, in case such information is available. We show, not surprisingly, that the prior knowledge about the diagonal structure of the signal covariance matrix can be exploited by the estimator so as to reduce the estimation errors, and to allow the array to cope with more signals. Interestingly enough, it will be shown that the number of signals that can be detected and localized by the array may well exceed the number of sensors. In fact, unlike the augmentation technique [6] which uses specially structured linear arrays to obtain such an improvement in the detection capability, our technique enables to obtain this improvement with arbitrary arrays.

Our solutions to both problems, i.e. colored noise and uncorrelated signals, are based on what in the statistical literature is known as the Generalized Least Squares (GLS) estimator [1]. The resulting estimator is proved to be both consistent and efficient, i.e. it asymptotically attains the Cramer-Rao lower Bound(CRB).

2 Problem Formulation

Consider q wave-fronts arriving from sources located at $\theta_1, \ldots, \theta_q$, and impinging on an array consisting of p sensors. For simplicity, assume that the sensors and the sources are all located on the same plane, and that the sources are in the far-field of the array, so that the wave-fronts are planar and $\{\theta_i\}$ represent their Directions-Of-Arrival(DOAs). Assume also that the sources emit narrow-band signals all centered around a common frequency. Let $s_i(t)$ denote the complex envelope of the i-th source signal, and let $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_p(t))^T$ denote the vector of complex envelopes formed from the signals received by the sensors, with T denoting transposition. In the presence of additive noise, this received vec-

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tor can be expressed as: $\mathbf{x}(t) = \sum_{k=1}^{q} \mathbf{a}(\theta_k)s_k(t) + \mathbf{n}(t)$, where $\mathbf{a}(\theta)$ is the steering vector of the array expressing its complex response to a planar wavefront arriving from direction θ, and $\mathbf{n}(t)$ is a vector formed from the complex envelopes of the sensor noises. This expression can be written more compactly as:

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{n}(t) \qquad (1)$$

where $\boldsymbol{\theta} \stackrel{\text{def}}{=} (\theta_1 \dots \theta_q)^T$, $\mathbf{A}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} [\mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_q)]$, and $\mathbf{s}(t) \stackrel{\text{def}}{=} (s_1(t), \dots, s_q(t))^T$. We shall assume that the steering vectors $\{\mathbf{a}(\theta)\}$ are known for all $\theta \in \boldsymbol{\Theta}$, where $\boldsymbol{\Theta}$ denotes the field-of-view.

Let the array be sequentially sampled at m time instants $t_1, \dots, t_m$ with $\mathbf{X} \stackrel{\text{def}}{=} [\mathbf{x}(t_1), \dots, \mathbf{x}(t_m)]$ denoting a matrix formed from the samples. Now, the problem can be stated as follows: Given the data $\mathbf{X}$ – estimate the number of sources q and their directions $\boldsymbol{\theta}$.

To solve this problem, we make the following assumptions:

*A*1. The noise samples $\{\mathbf{n}(t_i)\}$ are statisticaly independent zero-mean complex-Gaussian vectors, with a covariance matrix $\boldsymbol{\Sigma}$ given by the following linear model

$$\boldsymbol{\Sigma} = \sigma_1\boldsymbol{\Sigma}_1 + \sigma_2\boldsymbol{\Sigma}_2 + \dots \sigma_r\boldsymbol{\Sigma}_r \qquad (2)$$

where $\{\boldsymbol{\Sigma}_i\}$ are known matrices, and $\boldsymbol{\sigma} \stackrel{\text{def}}{=} (\sigma_1, \dots, \sigma_r)$ is an unknown parameter-vector. With no substantial loss of generality, we shall further assume that $\boldsymbol{\sigma}$ is real and that $\{\boldsymbol{\Sigma}_i\}$ are Hermitian.

*A*2. The signal samples $\{\mathbf{s}(t_i)\}$ are statisticaly independent zero-mean complex-Gaussian vectors independent of the noise samples, and having an unknown Hermitian covariance matrix $\mathbf{P}$.

We also investigate the case where $\mathbf{P}$ is a-priori known to be diagonal. Notice that assumption $A2$ does not exclude the possibility of the signals being coherent.

A simple example where the noise model (2) is valid, is the case where the noises in different sensors are uncorrelated, and their power levels are unequal, so that the noise covariance matrix is given by $\boldsymbol{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_p)$ where $\{\sigma_i\}$ denote the unknown noise power levels. Notice that this structure fits into our formulation (2), with $\boldsymbol{\Sigma}_i = \mathbf{I}_{ii}$, where $\mathbf{I}_{ij}$ denotes the elementary matrix defined as the matrix which has a unity in its (i,j)-th position and zeros elsewhere.

Another example is the case of the so-called "ambient noise" where the noise is contributed by external sources. If the spatial distribution of the external sources can be regarded as "continuous", it can be shown [14] that under reasonable conditions the noise covariance can be modelled by (2), where $\{\boldsymbol{\Sigma}_i\}$ are known Hermitian matrices, and $\{\sigma_i\}$ are real unknowns.

Based on the above assumptions and using (1),(2), the covariance matrix of the received vector is given by

$$\begin{aligned} \mathbf{R}(\boldsymbol{\phi}) &= \mathbf{A}(\boldsymbol{\theta})\mathbf{P}\mathbf{A}^H(\boldsymbol{\theta}) + \boldsymbol{\Sigma} \qquad (3) \\ &= \mathbf{A}(\boldsymbol{\theta})\mathbf{P}\mathbf{A}^H(\boldsymbol{\theta}) + \sigma_1\boldsymbol{\Sigma}_1 + \sigma_2\boldsymbol{\Sigma}_2 + \dots + \sigma_r\boldsymbol{\Sigma}_r \end{aligned}$$

where $()^H$ denotes complex-conjugate transposition, and $\boldsymbol{\phi}$ represents a real vector formed from all the unknown parameters, i.e. $\boldsymbol{\phi} \stackrel{\text{def}}{=} (\boldsymbol{\theta}^T, \vec{\mathbf{P}}^T, \boldsymbol{\sigma}^T)^T$, with $\vec{\mathbf{P}}$ denoting a vector formed from the free real parameters in $\mathbf{P}$, i.e. the real and the imaginary parts of the upper triangle entries of $\mathbf{P}$.

From assumptions $A1$ and $A2$ it follows that the Maximum Likelihood Estimator (MLE) is given by:

$$\begin{aligned} \hat{\boldsymbol{\phi}}_{MLE} &= \arg\max_{\boldsymbol{\phi}} \{L(\boldsymbol{\phi})\} \\ L(\boldsymbol{\phi}) &= -m\left[\log|\mathbf{R}| + \text{tr}(\mathbf{R}^{-1}\hat{\mathbf{R}}) + p\log\pi\right] \qquad (4) \end{aligned}$$

where $|\cdot|$ denotes determinant, tr() denotes trace, $\hat{\mathbf{R}} \stackrel{\text{def}}{=} \mathbf{X}\mathbf{X}^H/m$ is the sample-covariance matrix, and $\mathbf{R}$ is the array covariance matrix given in (3). Unfortunately, except for a few special cases, it is impossible to reduce the maximization dimesionality to q by eliminating $\mathbf{P}$ and $\boldsymbol{\sigma}$ analytically [2] [14]. Therefore, we use a different approach.

3 The GLS Estimator

The basic idea behind our approach is to select those parameters $\hat{\boldsymbol{\phi}}$ that give the "best fit" between the sample-covariance $\hat{\mathbf{R}}$ and the model-covariance $\mathbf{R}(\boldsymbol{\phi}) = \mathbf{A}(\boldsymbol{\theta})\mathbf{P}\mathbf{A}^H(\boldsymbol{\theta}) + \boldsymbol{\Sigma}$, with the goodness-of-fit criterion being

$$\ell(\boldsymbol{\phi}) = \frac{m}{2}\|\mathbf{I} - \hat{\mathbf{R}}^{-1/2}\left[\mathbf{A}(\boldsymbol{\theta})\mathbf{P}\mathbf{A}^H(\boldsymbol{\theta}) + \boldsymbol{\Sigma}\right]\hat{\mathbf{R}}^{-1/2}\|_F^2 \qquad (5)$$

where $\|\cdot\|_F^2$ denotes the Frobenius norm. The estimator obtained by minimizing this criterion, i.e.

$$\hat{\boldsymbol{\phi}}_{GLS} = \arg\min_{\boldsymbol{\phi}} \{\ell(\boldsymbol{\phi})\}$$

is referred to in the statistical literature as the Generalized Least Squares(GLS) estimator [1].

Since (5) is quadratic in the elements of $\mathbf{P}$ and $\boldsymbol{\sigma}$, analytical minimization with respect to (w.r.t.) $\mathbf{P}$ and $\boldsymbol{\sigma}$ is possible. First, we minimize with respect to $\mathbf{P}$ while holding $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}$ fixed. We then substitute the minimizing value $\hat{\mathbf{P}}(\boldsymbol{\theta}, \boldsymbol{\sigma})$ back into the cost function $\ell(\boldsymbol{\phi})$, and minimize with respect to $\boldsymbol{\sigma}$, thus obtaining an expression for the cost function that is a function of $\boldsymbol{\theta}$ only. The result is [?]

$$\hat{\theta}_{GLS} = \min_{\boldsymbol{\theta}}\left\{\ell(\boldsymbol{\theta}) = \frac{m}{2}[-\mathrm{g}^T\mathbf{G}^{-1}\mathrm{g}]\right\}$$

with $\mathbf{G}$ being a matrix and $\mathbf{g}$ a vector whose entries are given by $G_{ij} = \mathrm{tr}(\tilde{\mathbf{\Sigma}}_i\tilde{\mathbf{\Sigma}}_j) - \mathrm{tr}(\mathbf{P}_{\tilde{\mathbf{A}}(\boldsymbol{\theta})}\tilde{\mathbf{\Sigma}}_i\mathbf{P}_{\tilde{\mathbf{A}}(\boldsymbol{\theta})}\tilde{\mathbf{\Sigma}}_j)$ and $g_i = \mathrm{tr}[(\mathbf{I}-\mathbf{P}_{\tilde{\mathbf{A}}(\boldsymbol{\theta})})\tilde{\mathbf{\Sigma}}_i]$, where $\tilde{\mathbf{\Sigma}}_i \stackrel{\mathrm{def}}{=} \hat{\mathbf{R}}^{-1/2}\mathbf{\Sigma}_i\hat{\mathbf{R}}^{-1/2}$, and $\mathbf{P}_{\tilde{\mathbf{A}}(\boldsymbol{\theta})} \stackrel{\mathrm{def}}{=} \tilde{\mathbf{A}}(\tilde{\mathbf{A}}^H\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{A}}^H$ with $\tilde{\mathbf{A}}(\boldsymbol{\theta}) \stackrel{\mathrm{def}}{=} \hat{\mathbf{R}}^{-1/2}\mathbf{A}(\boldsymbol{\theta})$.

Similarly, for the case where $\mathbf{P}$ is a-priorily known to be diagonal we get

$$\hat{\theta}_{GLS} = \min_{\boldsymbol{\theta}}\{\ell(\boldsymbol{\theta})\}$$

$$\ell(\boldsymbol{\theta}) = -\frac{m}{2}(\mathbf{u}-\mathbf{W}\mathbf{V}^{-1}\mathbf{v})^T[\mathbf{U}-\mathbf{W}\mathbf{V}^{-1}\mathbf{W}^T]^{-1}(\mathbf{u}-\mathbf{W}\mathbf{V}^{-1}\mathbf{v})$$

where $\mathbf{u}$ and $\mathbf{v}$ are vectors whose entries are given by $u_i = \|\tilde{\mathbf{a}}(\theta_i)\|^2$; $i = 1,\ldots,q$, $v_i = \mathrm{tr}(\tilde{\mathbf{\Sigma}}_i)$; $i = 1,\ldots,r$, with $\tilde{\mathbf{a}}(\theta_i) \stackrel{\mathrm{def}}{=} \hat{\mathbf{R}}^{-1/2}\mathbf{a}(\theta_i)$, and where $\mathbf{U}, \mathbf{V}$, and $\mathbf{W}$ are matrices whose elements are given by $U_{ij} = |\tilde{\mathbf{a}}^H(\theta_i)\tilde{\mathbf{a}}(\theta_j)|^2$; $i,j = 1,\ldots,q$, $V_{ij} = \mathrm{tr}(\tilde{\mathbf{\Sigma}}_i\tilde{\mathbf{\Sigma}}_j)$; $i,j = 1,\ldots,r$, $W_{ij} = \tilde{\mathbf{a}}^H(\theta_i)\tilde{\mathbf{\Sigma}}_j\tilde{\mathbf{a}}(\theta_i)$; $i = 1,\ldots,q$, $j = 1,\ldots,r$.

We have thus reduced the problem dimensionality to q. The minimization can be efficiently accomplished by any multidimensional minimization technique. The estimators are proved to be consistent and asymptotically efficient [8].

A necessary condition for the uniqueness of the solution is $p^2 \geq q^2 + q + r$. For the case of uncorrelated signals the condition is given by $p^2 \geq 2q + r$. Notice that unlike the former condition, the later does not restrict the number of signals q to be less than the number of sensors p.

4 Simulation results

To demonstrate the performance of the proposed estimators we present several simulated experiments, all conducted with a 5-omnidirectional-element uniform circular array.

In the first experiment we compare the proposed estimator with existing estimator in the case of white noise. The simulated scenario consisted of two equipower coherent Gaussian sources with 10db SNRs located at 100° and 120°. The phase difference between the signals at the array center was 90°. The array diameter was 0.6λ, leading to a 3-db beamwidth of 26°, approximately. A set of 100 monte-carlo runs was carried out for each value of the number of samples, m, with the DOAs estimated in each run and the RMS DOA error (averaged over the two sources) computed from the whole set. The results obtained by using the GLS, the Weighted Subspace Fitting(WSF) [11], the Deterministic ML(DML) [12], and the ML estimators, along with the CRB (derived in [8]) are displayed in Fig.1. Clearly, the difference in the performance of these estimators is marginal.

In a second experiment we demonstrate the performance for unequal noise power levels at the various sensors. The noise covariance was $\mathbf{\Sigma} = \sigma^2\mathrm{diag}(1,2,4,2,1)$,

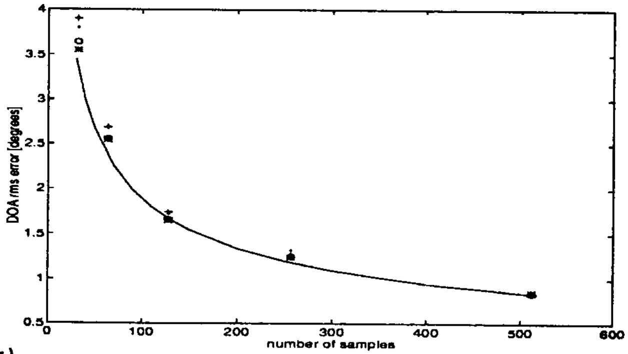

Figure 1: The RMS DOA error for two equipower coherent sources located at 100° and 120° impinging on a five element uniform circular array with 0.6λ diameter. Spatial noise is white and SNR is 10db. The solid line represents the CRB. The $+, 0, *$ and $\cdot$ represent the results of the GLS, WSF, ML, and DML estimators, respectively.

where σ^2 was chosen so that the SNR measured at the first sensor is 10db. The GLS estimator, using the noise model $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1,\ldots,\sigma_5)$, was compared with the WSF, the ML, and the DML estimators all using a (incorrect) spatially-white noise model. The results are shown in Fig.2 and demonstrate the advantage in using the GLS estimator in this case.

To demonstrate the advantage in exploiting the fact that the sourced are uncorrelated we show in Fig.3 the CRB for 4 equipower uncorrelated sources located at 100°, 120°, 140°, 160°. The noise is spatially and temporally white, and the number of snapshots is $m = 256$. The dashed line represents the CRB for the first source DOA error at case the prior knowledge about the lack of correlation is exploited, whereas the solid line represents the CRB at case this prior knowledge is not exploited. Notice that the performance difference is more conspicous at low SNR.

Next, we demonstrate the capability of the array to localize uncorrelated signals even if their number exceeds the number of sensors. The scenario consisted of 6 equipower uncorrelated sources located at 50°, 100°, ..., 300°. The array diameter was increased to 1.5λ, and the noise was spatially and temporally white. Fig.4 Displays the CRB for the DOA error of the first source. Notice that, in contrast to the case where the number of sensors exceeds the number of sources, the error here, interestingly enough, does not vanish at high SNR values. (However, it is guaranteed to vanish for high values of m at a rate of $m^{-1/2}$).

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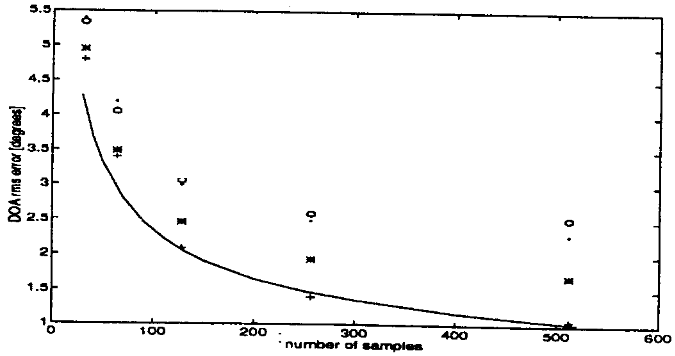

Figure 2: Same scenario as in the previous figure except that here the spatial noise is non-white, with $\Sigma = \text{diag}(1, 2, 4, 2, 1)$.

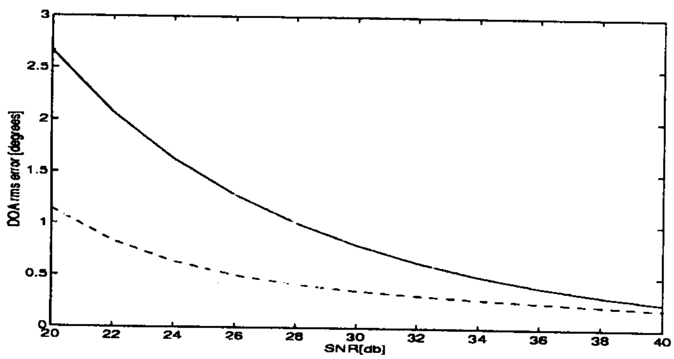

Figure 3: Four equipower uncorrelated sources located at 100°, 120°, 140°, and 160° impinging on a five element uniform circular array with 0.6λ diameter. Spatial noise is white, and the number of samples is 256. The dashed line represents the CRB for the first source for the case where the sources are a-priori known to be uncorrelated, whereas the solid line represents the case where this knowledge is not exploited.

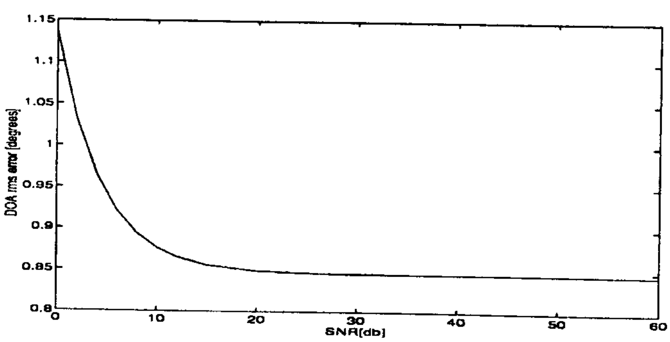

Figure 4: Six equipower uncorrelated sources located at 50°, 100°, 150°, ..., 300°, impinging on a five element uniform circular array with 1.5λ diameter. Spatial noise is white, and the number of samples is 256. Shown is the CRB for the first source.

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