EXTENSION OF THE PISARENKO METHOD TO SPARSE LINEAR ARRAYS.

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ABSTRACT

When applied to array processing, the Pisarenko harmonic retrieval method is limited to linear equispaced arrays. We present an approach that allows to extend it to general arrays. For the ease of exposition, we consider only sparse linear arrays. Though limited in generality, these arrays already permit to localize up to $N(N-1)/2$ narrow-band sources with N-sensors. We actually show that the Pisarenko approach can be seen as a deconvolution or model-fitting approach that minimizes an ℓ_1 norm and can be implemented as a standard linear program. Further extensions allowing to model and take into account the statistical nature of the data (the estimation errors) are also proposed.

1. INTRODUCTION

The covariance matrix of the outputs of a linear equispaced array of N sensors is hermitian and Toeplitz. It has $2N-1$ real degrees of freedom and one can localise up to $N-1$ independent narrowband far-field sources in additive white noise. If the array is linear but no longer equispaced, the covariance matrix is hermitian with a constant diagonal, it can have up to $L = N(N-1)+1$ real degrees of freedom and provided the array manifold is unambiguous, one can locate up to $N(N-1)/2$ sources.

While for linear equispaced arrays, there are parametric means (ME methods, Pisarenko, min-norm, root-MUSIC..) to achieve the localization, for non-equispaced arrays no such systematic approach exists and some kind of search has to be performed. MUSIC can be applied to such arrays if the number of sources is smaller than the number of sensors, but MUSIC does not take into account the structure of the covariance matrix that stems from the array geometry in the estimation of the signal subspace. The conventional beamformer (CBF) can handle such arrays without limitation on the number of sources. It uses the information about the structure of the array in the definition of the steering vector but its performances are poor for closely spaced sources.

We apply a model-fitting algorithm to the output of a CBF that drastically improves these performances. In [3] such a deconvolution approach has been applied locally to spatial sectors, here the approach is global and can be interpreted as an extension of the Pisarenko method to non-equispaced arrays. It is based on the same premises as the methods proposed in [1] and belongs to the correlation matching spectral estimation schemes introduced in [2]. Its implementation relies on a standard linear programming algorithm.

2. THE MODEL

We consider a linear array having N sensors in position $[0, x_1, x_2, ..x_{N-1}]$. The bearings ϕ are measured with respect to broadside. We take as unit of length $\lambda/2$ and define the spatial frequencies ν as given by $\nu = (\sin\phi)/2$. The steering-vector associated with spatial frequency ν admits then the following model :

$$d\theta(\nu) = \left[1 \; e^{2i\pi\nu x_1} \; e^{2i\pi\nu x_2} \; \dots \; e^{2i\pi\nu x_{(N-1)}}\right]^T \qquad (1)$$

For a scenario with P uncorrelated sources the exact covariance matrix R_s of the outputs of the array is then the sum of P dyads $d\theta(\nu_.)d\theta(\nu_.)^*$ weighted by the source powers $a_.$:

$$R = \sum_{p=1}^{P} a_p d\theta\,(\nu_p)\, d\theta(\nu_p)^* \qquad (2)$$

In the presence of additive white noise with variance σ_n^2, one has :

$$R = R_s + \sigma_n^2 I \qquad (3)$$

This matrix is hermitian with constant diagonal and element (k,l) $(k \neq l)$ equal to :

$$r_{k,l} = \sum_{p=1}^{P} a_p e^{2i\pi\nu_p\delta_{k,l}} \qquad (4)$$

where $\delta_{k,l} = x_k - x_l$ denotes the inter-sensor distance. If all the elements of $\{\delta_{k,l}, k > l\}$, the so-called co-array [4,6], are distinct, the matrix has $L = N(N-1)+1$ real degrees of freedom and one can associate to it a L-dimensional real vector.

The beamformer output at spatial frequency f is :

$$y(f) = \frac{1}{N^2}\, d\theta^*(f) R d\theta(f) \; . \qquad (5)$$

For R as in (3), it can be decomposed as :

$$y(f) = \sum_{p=1}^{P} a_p F_0(f-\nu_p) + \frac{1}{N}\,\sigma_n^2 \qquad (6)$$

where :

$$F_0(f-\nu) = \frac{1}{N^2}|d\theta(f)^* d\theta(\nu)|^2 = \frac{1}{N^2}|d\theta(0)^* d\theta(f-\nu)|^2 \quad (7)$$

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is the output of the beamformer at spatial frequency f for a source with unit power located at spatial frequency ν. For an equispaced linear array, $F_\nu(f)$ is a shifted version of the discrete Fejer kernel, the Fourier transform of the sampled *triangle*. Note that the shift property holds even for non-equispaced linear arrays.

If the covariance matrix has L real degrees of freedom, we propose to take as observations L equispaced samples of the beamformer output :

$$y(f_k) \text{ for } f_k = \frac{k}{L}\ ,\ \ k = 0, \pm 1, \pm 2, \ldots \pm \frac{L-1}{2} \qquad (8)$$

Provided the array is unambiguous, these outputs contain all the information available. The deconvolution or model-fitting approach we propose to implement amounts then to fit to these observations, a weighted sum of a large number, say M, of similarly sampled regularly shifted version of the kernel (7) and to evaluate the weights. In matrix notations, let b be the L-dimensional observation-vector and define A, the (L, M) matrix having as columns the sampled and regularly shifted kernels. The difficulty is then to single out among the $(M-L)$-dimensional affine sub-space of solutions to $AX = b$, the solution (weights) X that corresponds to the underlying scenario. Since the number P of sources is quite small ($P \ll M$), one seeks a parsimonious solution X with positive components. The answer we will arrive to, is actually quite close to solving the following optimization problem :

$$\text{Min } \|X\|_1 \quad \text{subject to} \quad AX = b\ ,\ X \geq 0 \qquad (9)$$

in which the noise contribution is not yet taken care of. It is important to note that all the columns of A should have the same ℓ_1-norm.

3. THE SET OF COVARIANCE MATRICES

The covariance matrix R of the output of an arbitrary linear array of N sensors can be decomposed into the contribution R_s of the sources (2), and the white noise contribution $\sigma_n^2 I$. Relation (2) says that R_s is a positive linear combination of dyads. This implies that R_s belongs to the closed convex conical hull of the set of the dyads. This is a closed convex cone with vertex at the origin whose generating elements are the *dyads*. Since the contribution of the additive white noise can, for instance, be constructed as a convex combination of L equispaced dyads, it clearly belongs to the interior of the same cone. The complete matrix R thus always lies in the interior of the cone. The interior and the boundary of the cone are quite difficult to define in terms of the covariance matrix except for very specific sparse linear arrays [4]. The elements on the boundary of the cone include, but are not limited to, rank deficient matrices of the form (2).

To each dyad, one can associate a L-dimensional real vector. The same holds for the covariance matrices R and one can think of this cone as lying in an L-dimensional subspace of the N^2-dimensional real vector space associated with the hermitian matrices of order N.

Since the cone is convex, any matrix of the cone can be represented as a convex combination of L or less independent dyads [2], the correlation matching procedure consists in trying to obtain such a representation for the observed covariance matrix. It is in general non-unique and each possible representation is associated with a different extension of the covariances. However in the case we consider, where the covariance corresponds to the output of an array and the sources are assumed to be independent point-sources in the far-field, a very specific decomposition is sought for.

For P small enough, the contribution of the P dyads associated with the sources and the contribution of the white noise are indeed easy to distinguish. One is in the degenerate situation where the global source-contribution lies on the boundary of the cone. To identify it, one then simply maximizes the contribution of the noise. This forces the remainder to lie on the boundary.

Nothing guarantees however that this element on the boundary, which represents the global contribution of the P sources, has a unique decomposition into P dyads. But this is a different difficulty that concerns the ambiguity of the array manifold. Even in the case of linear (non-equispaced) arrays, the unicity of the decomposition is a difficult question.

4. IDENTIFIABILITY AND AMBIGUITY

One could also say : invertibility and uniqueness. Provided the array is un-ambiguous, by invertibility or identifiability, we mean the conditions under which the knowledge of R allows to reconstruct the underlying scenario. The investigation of (2,4) leads to the following conclusions.

- If $P < N$, the covariance matrix R (3) is rank deficient in the absence of noise and one can recover the scenario in a number of different ways, MUSIC for instance. It is easy to realize that the noise power is simply the value, of the possibly multiple, minimal eigenvalue of R.
- If $N < P \leq (L-1)/2$, the covariance matrix (3) is full rank even in the absence of noise and the task is less trivial. One can however check that that the number $2P+1$ of unknowns $(\{a_p, \nu_p\}_{p=1,..,P}, \sigma_n^2)$ is smaller than the number L of real degrees of freedom. Thus, if the array is *unambiguous*, it is possible - though less trivial- to identify the scenario. One has to find a solution to a system of L non-linear equations in $2P+1$ unknowns.
- If $P > (L-1)/2$, the task is impossible. The covariance matrix lies in the interior of the cone even in the absence of additive noise. The unique decomposition into the sum of an element on the boundary and the white noise element leads to an element on the boundary with no relation to the true scenario.

The maximal number of identifiable sources is thus $(L-1)/2$. This is not in contradiction with the generalization of a theorem of Caratheodory given in [2, appendix B] which, when applied to linear non-equispaced arrays, says that any covariance matrix on the boundary of the considered cone can be reconstructed as a convex combination of at most $(L-1)$ generating elements. This theorem only gives an upper-bound on the number of elements that is required. In the algorithm described below, once on the boundary we will indeed generically require $(L-1)$ dyads, but this will be the case because we discretize the spatial frequency domain, because we replace the original cone by a polyhedral cone. Each true dyad (source) is then interpolated by a couple of neighboring dyads (sources).

Little is known about conditions on the coarray that guarantee the uniqueness of the solution. Non-uniqueness or ambiguity means that an element R_s on the boundary of the cone (corresponding thus to $P \leq (L-1)/2$ in (2)) has not an unique decomposition into generating elements (dyads). Even for linear arrays, the study of ambiguity is a difficult task and only few sufficient conditions for non-ambiguity or necessary conditions for ambiguity are available. Ambiguity appears trivially for linear equispaced arrays with inter-sensor distance greater than $\lambda/2$. This corresponds to undersampling. Some specific unambiguous non-equispaced linear arrays are known as for instance the minimum redondancy linear arrays [4]. A non-trivially ambiguous non-equispaced linear arrays is analysed in [5].
In the simulations presented later, we will consider a minimum redondancy linear array. Such arrays are trivially non ambiguous and the number of identifiable sources is easily deduced from the cardinality of the co-array.

5. DEVELOPMENT

Note that fitting a model to the covariance matrix or to the outputs of the CBF is equivalent provided the two quantities are in a one-to-one relation. This is the case for a general non-equispaced un-ambiguous array if the number of evaluated beams is equal to the number L of real degrees of freedom of the covariance matrix. Therefore instead of matching the complex covariances (4), we will match the L real CBF outputs (8).
The description given in section 3. about the structure of the set of the covariance matrices can be transposed on the CBF-side : all potential CBF-outputs belong to the positive cone whose generating functions are $F_0(f-\nu)$ (7). And any CBF-output that corresponds to an invertible scenario can be represented as the sum of P weighted and shifted generating functions and the constant contribution of the noise (6). The generating function is the discrete Fourier transform of the weighted covariance sequence that is known only on the coarray.
Following these investigations upon the structure of the covariance matrices, we propose to modify algorithm (9) developed from a straight-forward model-fitting philosophy in which the noise contribution had not yet been taken into account.
We compute the output of the CBF at L equispaced spatial frequencies (8) and store these quantities in an L-dimensional real vector b. Our aim is to rewrite this vector as the sum (6) of a "white-noise" contribution of maximal amplitude and the smallest possible number of similarly-discretized frequency-shifted weighted CBF-kernels (7). This amounts to find a solution of a highly non-linear system of L equations. Though the solution we seek is well defined and generally unique, it is difficult to obtain and would require an initialization procedure.
We thus decide to discretize the spatial frequencies (bearings) i.e. assume that the sources can only be located at the M nodes of a grid in the spatial frequency range $]-\frac{1}{2},+\frac{1}{2}[$. This amounts to replace the original cone by a polyhedral cone defined by regularly spaced basis elements. We also explicitly add to these basis elements an element representing the noise contribution. On the CBF-side, the contribution of the noise is a constant, the same on all the beams. Maximizing this constant pushes the remainder on the boundary of the polyhedral cone which coincides with the boundary of the original cone only at the nodes. A sufficiently dense uniform sampling of the spectral support allows to approximate the original cone and its boundary to arbitrary precision and to reduce the bias at will.
This implementation of the decomposition problem leads to a standard linear program, quite similar to (9), for which efficient algorithms (e.g. the simplex method) that converge to a global optimum are available :

$$\text{Max } v \quad \text{subject to :} \quad A\,X + v\mathbf{1} = b \;,\; X \geq 0,\; v \geq 0 \quad (10)$$

where : b is an L-dimensional vector containing the outputs of the CBF (8), A is an (L, M) matrix ($M \gg L$) the columns of which contain the shifted and sampled generating functions (7), 1 is a column-vector of one's modeling the white noise contribution and v models the noise variance.

Some remarks are in order.
• If there are feasible points, the problem has a solution. There is always a solution, say (X^*, v^*) at a vertex of the domain. This means that there is a solution with at most L strictly positive components. Since v^* is always (strictly) positive, there are at most $L-1$ components of the M-dimensional (weighting) vector X^* that are positive.
• Due to the discretization of the spatial frequencies (bearings) with a step-size $\frac{1}{M}$, a single source will systematically be reconstructed by the two neighboring nodes. $L-1$ positive components in X^* thus allow to localize at most $(L-1)/2$ sources. This is also the maximal number of admissible sources for invertibility reasons. (see section 4.)
• The spatial frequency to be attributed to a source associated with two such positive X^*-components is easily obtained from an interpolation of the "indices" of the components. The amplitude is obtained from the values of the components.
• Strictly speaking, the 2 neighboring elements never allow to exactly reconstruct a missing generating function, other elements (distributed over the domain) contribute also. The amplitude of these further contributing elements can be monitored by choosing the discretization step $\frac{1}{M}$. This step should also be chosen in accordance with the accuracy one expects or seeks from the array.
• The solution (X^*, v^*) may have less than L strictly positive components. Such a solution is termed degenerate, but standard softwares can handle this case. Remember that the Pisarenko approach has (also) some difficulties in handling such situations where the number of sources is less than the maximum admissible.
• In practice only an estimate $\hat{b}$ of b is available. The algorithm will then generically use the maximal number $(L-1)/2$ of sources in the reconstruction. The problem then is to decide whether a source with low amplitude is indeed a source or is induced by the noise, the estimation error and the discretization step. Deciding how many sources there are, or which sources are true ones, is a detection problem. An *a priori* evaluation of the detection threshold of the array should thus be performed and allow to take a decision by looking at the amplitudes.

6. SIMULATION RESULTS

We apply the approach described above to simulated data. Thus only an estimate of R, obtained from T observations (the snapshots), is available : $\hat{R} = \frac{1}{T}\sum_{k=1}^{T} X_k X_k^*$. The "exact" representation defined above is thus perturbed by the estimation error due to the finite number of observations. One should no longer seek an exact solution of the set of equations (10) and introduce a distance taking into account the statistical properties of the observations.

We consider a minimum redondancy linear array with 4 sensors [4][1]. The sensors are at positions $\{0, 1, 4, 6\}$ (half wavelengths) and the exact covariance matrix R has $L = 13$ real degrees of freedom. An estimate $\hat{R}$ is obtained from $T = 100$ snapshots. It is used to evaluate the CBF at 13 equispaced bearings (8) and to form $\hat{b}$. We take a matrix A in (10) with 13 rows and $M = 143$ columns which corresponds to a discretisation step in spatial frequency of $\frac{1}{M} \simeq 7.10^{-3}$. The sensitivity with respect to this parameter is quite low : taking $M = 91$ leads to strictly similar results but the spurious sources get stronger though always smaller in amplitude than the true sources. The scenario consists of 2 sources that have a common SNR of 0 dB with respect to the white noise ($a_1 = a_2 = 1$ and $\sigma_n^2 = 1$ in (2), (3)). We simulated 2 different situations : -in one case the 2 sources are at spatial frequencies 0 and .1 (0 and 11.5 degrees) and -in the second case the spatial frequencies are 0 and .06 (0 and 6.9 degrees). The means and standard deviations of the estimated spatial frequencies, obtained from 50 independent trials, are given in row 5 and 6 of **Table 1** below.

source number	1	2	1	2
source power in dB	0	0	0	0
spatial frequency	0	.1	0	.06
bearing in degrees	0	11.5	0	6.9
estimated spatial freq.	.0006	.0999	-.018	.061
standard deviation	.0126	.0141	.061	.019
estimated spatial freq.	.0015	.1012	.0009	.0615
standard deviation	.0067	.0057	.0104	.0118

Table 1: Results over 50 trials for 2 two-sources secnarios

In the last 2 rows, we give the results obtained on strictly the same data with a slightly different algorithm, closer to (9) that takes into account the existence of estimation errors. The improvment is most sensitive in the standard deviations that are quite smaller and thus comparable to the Cramer-Rao bounds. The linear programming algorithm used is taken from the NAG-library and the initial weighting vector X (in (10)) is taken equal to 0.

The covariance matrix R of the array we consider, contains indeed the covariances $\{r_k, k = 0, 1, .., 6\}$. This means that with $\hat{R}$, one could built an order 6 Toeplitz covariance matrix and apply standard localization algorithms. This is absolutly not a necessity for our approach and has been chosen to ease comparisons and to be sure that the array presents no ambiguity. The reader can check with Table 1 that other algorithms have quite poorer performances.
If despite the presence of the noise, the vector associated with the estimated covariance matrix is not in the interior of the cone the program (10) has no feasible solution. To avoid this situation we always add a positive constant to $\hat{b}$, i.e. we replace the constraint $AX + v1 = \hat{b}$ by $AX + v1 = \hat{b} + \alpha 1$. This does not modify the solution X^*.

7. CONCLUSIONS AND EXTENSIONS

Several modifications of the approach or extensions to other problems can be developed. As far as the localization problem considered here, is concerned, one should introduce the statistical properties of the observations and define a distance to be minimised or a likelihood to be maximised. The standard statistical model used in the array processing context leads to a CBF-output vector $\hat{b}$ that is asymptotically, in the number T of snapshots, gaussian with a covariance matrix that can be estimated from $\hat{R}$. Returning to the model-fitting or deconvolution interpretation (9) of the algorithm, one can modify the problem formulation in several different ways to allow the linear programming procedure to use this information. The results of one of these modifications are presented at the bottom in Table 1. In some sense, one can identify part of the noise realisation in the CBF-outputs $\hat{b}$. This allows also to handle the situation where the noise sub-space has dimension greater than one [7], not generally considered in the Pisarenko approach and amounts to substract more than just $\lambda_{min} I$ from the estimated covariance matrix.

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