AN SNR THRESHOLD FOR 2-D DAMPED HARMONIC MODELS

Michael P. Clark

Air Force Institute of Technology
AFIT/ENG; 2950 P. Street
Wright-Patterson AFB, OH 45433-7765

ABSTRACT

Consider the problem of finding a lower bound on the signal to noise ratio (SNR) at which any unbiased 2-D harmonic estimation scheme may *resolve* closely spaced parameters. In particular, consider the Cramer-Rao bounds associated with the parameters. If the parameter estimates are unbiased and normally distributed, the bounds give the smallest possible ellipsoid about each of the parameters which contains a given amount of probability mass. These ellipsoids increase in size with decreasing SNR. This paper presents a method for determining the largest SNR at which any two of the ellipsoids touch. This is deemed the resolution threshold SNR.

1. THE 2-D MODEL

Consider 2-D data of the form $y(m,n)$, where $0 \leq m \leq M-1$, and $0 \leq n \leq N-1$. Suppose the data have a 2-D modal structure,

$$y(m,n) = \sum_{i=1}^{p} s_i \lambda_i^m \gamma_i^n + w(m,n), \qquad (1)$$

where p is the number of modes and $w(m,n)$ is complex, additive white Gaussian noise of variance σ^2. Each 2-D mode is parameterized by the real parameter vector

$$\boldsymbol{\theta}_i \triangleq [\text{Re}(\lambda_i),\ \text{Im}(\lambda_i),\ \text{Re}(\gamma_i),\ \text{Im}(\gamma_i)]^T, \qquad (2)$$

where each λ_i is a complex modal parameter for the first dimension, and each γ_i, a complex modal parameter for the second dimension. The complex number s_i denotes the amplitude of the i^{th} 2-D modal signal.

Recent research into parametric techniques for identifying the parameters of this above model has been extensive. At high SNRs these techniques are preferable to classical nonparametric spectrum estimators in that they are high-resolution. That is, they offer the possibility of resolving parameters whose spacings are smaller than the Rayleigh limit. However, this is possible only when the variances of the parameter estimates are small in comparison to the parameter spacings. Since the variances of the estimates increase with decreasing SNR, one may define a resolution threshold noise level. This is the noise level at which the variances of the estimates are deemed to be commensurate with parameter spacings. For one-dimensional parameter estimation, definitions of resolvability closely related to the above were proposed by Oh and Kashyap [1] and Yau and Bresler [2].

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2. A RESOLUTION METRIC

To define resolution for the 2-D parameter estimation problem, one may rely on the classical notion of a spectrum analysis technique resolving two closely spaced lines when the estimated spectrum exhibits two distinct peaks. In the parametric case, parameters may be considered resolvable when a scatter plot of the parameters yields *distinct* clusters of estimates. Figure 1 shows such a scatter plot for scenario with $N = M = 5$, $p = 2$, $\boldsymbol{s} = [1\ 1]^T$, $\boldsymbol{\lambda} = [e^{j0.1\pi}\ e^{-j0.2\pi}]^T$, $\boldsymbol{\gamma} = [e^{j0.2\pi}\ e^{j0.3\pi}]^T$, and SNR = 15dB. The method used to estimate the parameters is described in [3].

The purpose of this paper is to address questions about the resolution of 2-D parameters. To accomplish this task, each parameter estimate is assumed to be unbiased and normally distributed. Specifically, suppose that each estimate is distributed as $\hat{\boldsymbol{\theta}}_k : \text{N}[\boldsymbol{\theta}_k, C_k]$, where $C_k \in \mathbb{R}^{d\times d}$. That is, suppose that the density function for each $\hat{\boldsymbol{\theta}}_k$ is

$$\begin{aligned} f(\hat{\boldsymbol{\theta}}_k) &= (2\pi)^{-\frac{d}{2}} \det(C_k)^{-\frac{1}{2}} \\ &\cdot \exp\left(-\frac{1}{2}(\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k)^T C_k^{-1} (\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k)\right), \end{aligned} \qquad (3)$$

where the covariance matrix is of the form $C_k \triangleq \sigma^2 R_k$, σ^2 represents the noise variance, and the estimates are

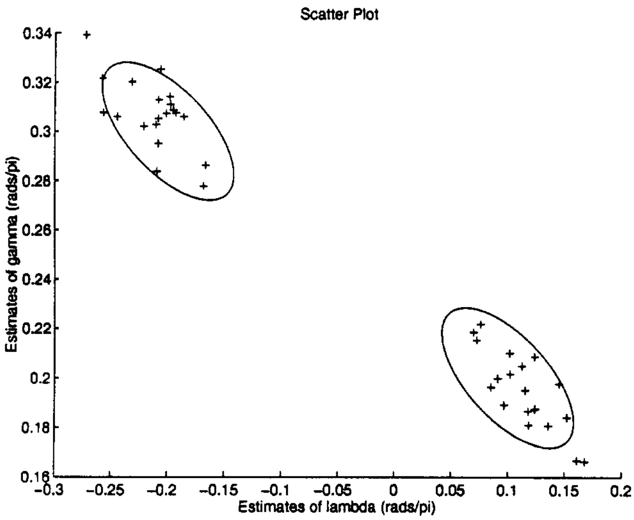

Figure 1: Scatter Plot with CRB Ellipses Containing 90% Probability Mass

unbiased. (The assumption of unbiasedness is justified for many techniques [3, 4, 5]).

The level curves associated with the densities of the estimates are the ellipsoids

$$\mathcal{E}_k \triangleq \{\boldsymbol{\theta} : (\boldsymbol{\theta}-\boldsymbol{\theta}_k)^T C_k^{-1}(\boldsymbol{\theta}-\boldsymbol{\theta}_k) \leq r^2\}; \quad 1 \leq k \leq p \quad (4)$$

where r^2 is some nonnegative constant. Here r is chosen so that each ellipsoid encloses some arbitrarily large fraction of the probability mass. Consider plotting the ellipsoids associated with each of the parameters in a common coordinate system.

Of course, for unbiased estimates, the covariance matrices are bounded by the 2-D Cramer-Rao bound (CRB) matrices: $R_k \geq Q_k$, where $\sigma^2 Q_k$ is the CRB matrix for the parameter $\boldsymbol{\theta}_k$. That is, the probability mass enclosed by the CRB ellipsoids is a subset of the mass enclosed by the ellipsoids of any unbiased estimator. The smallest signal to noise ratio (SNR) at which the ellipsoids corresponding to the CRB's are disjoint is deemed the *resolution threshold SNR*. Such a threshold may be of use in answering questions about the applicability of parametric estimators in low SNR environments. In particular, if the SNR is below threshold, parametric methods will not necessarily offer resolution superior to nonparametric approaches.

Figure 1 shows the 90% error ellipses obtained using the CR bounds of [6]. Note that since they are disjoint, it is likely that 15 dB is above the resolution threshold. However, one cannot be certain since this figure treats only two of four possible real parameters and does so in a polar coordinate system. While a polar coordinate system is convenient for representing the 2-D modes, it is the wrong topological space in which to phrase the resolution problem (because, for instance, the phase angles are periodic). For this reason this paper considers only the rectangular representation of equation 2.

3. FINDING THE THRESHOLD SNR

An equivalent definition for the resolution threshold is the smallest noise level at which any two ellipsoids corresponding to the CR bounds become tangent. Formally, the problem of finding the resolution threshold is as follows:

Problem 1 *Consider p unbiased estimators $\{\hat{\boldsymbol{\theta}}_k\}_{k=1}^p$ of some 4-dimensional, real parameters $\{\boldsymbol{\theta}_k\}_{k=1}^p$. Suppose that each $\hat{\boldsymbol{\theta}}_k$ is normally distributed with CRB matrix $\sigma^2 Q_k$. Suppose further that $\boldsymbol{\theta}_k \neq \boldsymbol{\theta}_l$ if $k \neq l$. Find the minimum value of σ^2 for which there exists a $\boldsymbol{\theta} \in \mathbb{R}^4$, and integers $1 \leq k, l \leq p$ with $k \neq l$, such that*

$$(\boldsymbol{\theta}-\boldsymbol{\theta}_k)^T Q_k^{-1}(\boldsymbol{\theta}-\boldsymbol{\theta}_k) \leq \sigma^2 r^2, \quad (5)$$
$$(\boldsymbol{\theta}-\boldsymbol{\theta}_l)^T Q_l^{-1}(\boldsymbol{\theta}-\boldsymbol{\theta}_l) \leq \sigma^2 r^2. \quad (6)$$

Here r is chosen so that each ellipsoid in $\{\boldsymbol{\theta} : (\boldsymbol{\theta}-\boldsymbol{\theta}_n)^T(\sigma^2 Q_n)^{-1}(\boldsymbol{\theta}-\boldsymbol{\theta}_n) \leq r^2\}_{n=1}^p$ encloses some given amount of probability mass, P. Thus the problem is to find the minimum value of σ^2 such that at least two ellipsoids are tangent with disjoint interiors

In [7, 8], the problem of finding the threshold SNR in spaces of arbitrary dimension is explored. The following provides the key theorem behind this method.

Define the two ellipsoids associated with the parameters $\boldsymbol{\theta}_i$ and $\boldsymbol{\theta}_j$ as

$$E_i \triangleq \{\boldsymbol{x} : (\boldsymbol{x}-\boldsymbol{\theta}_i)^T Q_i^{-1}(\boldsymbol{x}-\boldsymbol{\theta}_i) \leq \sigma^2 r^2\}, \quad (7)$$
$$E_j \triangleq \{\boldsymbol{y} : (\boldsymbol{y}-\boldsymbol{\theta}_j)^T Q_j^{-1}(\boldsymbol{y}-\boldsymbol{\theta}_j) \leq \sigma^2 r^2\}. \quad (8)$$

As Q_i^{-1} and Q_j^{-1} are positive definite by assumption, there exists a nonsingular matrix $T \in \mathbb{R}^{d\times d}$ such that

$$T^T Q_i^{-1} T = D_x, \quad T^T Q_j^{-1} T = I, \quad (9)$$

where $D_x \triangleq \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_d)$ is a positive definite matrix. T is said to simultaneously diagonalize Q_i^{-1} and Q_j^{-1} [9]. Let

$$\tilde{\boldsymbol{x}} \triangleq T^{-1}(\boldsymbol{x}-\boldsymbol{\theta}_i), \quad \tilde{\boldsymbol{y}}_0 \triangleq T^{-1}(\boldsymbol{\theta}_j-\boldsymbol{\theta}_i). \quad (10)$$

The following theorem then gives the key result.

Theorem 1 *Sufficient conditions for the ellipsoids E_i and E_j to be tangent with disjoint interiors are*

$$\tilde{\boldsymbol{x}}_* = (I-\alpha D_x)^{-1}\tilde{\boldsymbol{y}}_0, \quad (11)$$

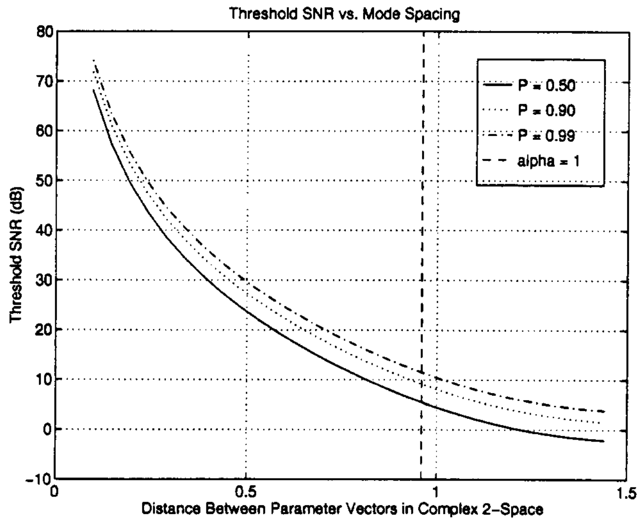

Figure 2: Threshold SNR versus the Distance Between the 2-D Modes

$$\sigma^2 r^2 = \tilde{x}_*^T D_x \tilde{x}_*, \quad (12)$$

$$\sigma^2 r^2 = (\tilde{x}_* - \tilde{y}_0)^T (\tilde{x}_* - \tilde{y}_0), \quad (13)$$

$$\alpha < 0. \quad (14)$$

It is shown that the solution exists and is unique. The solution for the threshold noise level is then found as

$$\sigma_T^2 = \frac{\tilde{y}_0^T D_x (I - \alpha D_x)^{-2} \tilde{y}_0}{r^2} \quad (15)$$

where α is the unique negative zero of the rational function

$$\tilde{y}_0^T D_x (I - \alpha D_x)^{-2} \tilde{y}_0 - \alpha^2 \tilde{y}_0^T D_x^2 (I - \alpha D_x)^{-2} \tilde{y}_0. \quad (16)$$

A fast algorithm for computing the coefficients of the numerator polynomial of this function is also given in [7, 8]. Further an algorithm which, given the covariance matrices for each of the multidimensional parameters, provides the threshold SNR is presented.

4. EXAMPLES

Of particular interest is the SNR threshold as a function of the distance between two closely spaced 2-D modes. For the earlier example such a plot is given by Figure 2. In this figure the second mode varies as

$$\begin{bmatrix} \tilde{\lambda}_2 \\ \tilde{\gamma}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \gamma_1 \end{bmatrix} + \alpha \begin{bmatrix} \lambda_2 - \lambda_1 \\ \gamma_2 - \gamma_1 \end{bmatrix}, \quad (17)$$

with $\alpha \in (0, 1.5]$ and the distance between modes measured as $d \triangleq \|[\tilde{\lambda}_2 - \lambda_1 \; \tilde{\gamma}_2 - \gamma_1]\|_2$. This plot shows that the SNR threshold increases rapidly with decreasing parameter spacing. Further, it shows that the threshold is not as sensitive to the choice of P as one might suspect. Next, notice that for our earlier example (for which $\alpha = 1$) that the distance between the modes is approximately 0.96 and the SNR threshold is 9.13 dB for the case $P = 0.9$. Finally, to show that the ellipsoids are indeed tangent at the threshold SNR see Figure 3. This figure shows a 2-D slice of the 4-D ellipsoids in the plane containing their centers and their point of tangency.

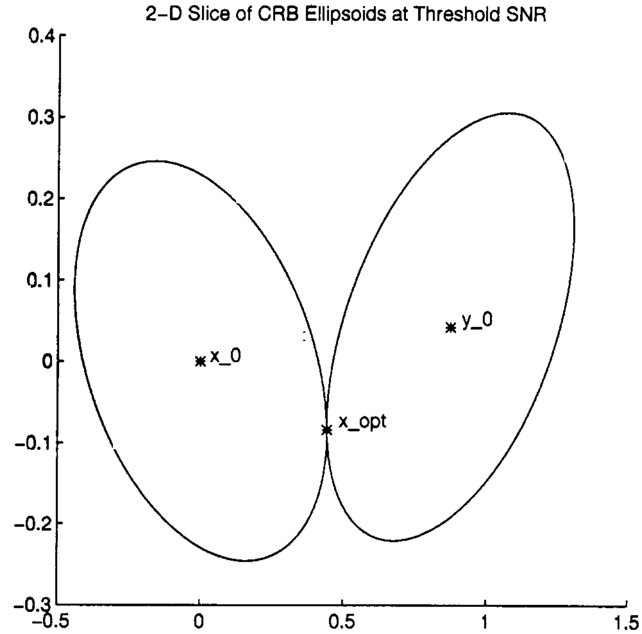

Figure 3: 2-D Slices of CRB Ellipses at the Threshold SNR of 9.13 dB

The next step is to observe the mean squared error (MSE) of a parametric technique and see if it exhibits threshold behavior at the computed SNR threshold. Toward this end, the sample MSE based on 100 different noise realizations was computed at many different SNRs. This was done for each of two estimation methods. The first was the parametric technique of [3] (2D IQML), and the second, a simple nonparametric technique. This second technique merely estimated $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ by finding the 2 bins of greatest modulus in the 2-D discrete Fourier transform (DFT). Note that for the case of interest the parameters of $\boldsymbol{\gamma}$ lie in a single Fourier bin. Figures 4 and 5 show the MSEs for each of the phases of the parameters in $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$. They also provide the CRBs for each of parameters.

Notice that the parametric technique exhibits threshold behavior at an SNR above the predicted threshold. This is precisely as expected since the threshold was found using ellipsoids corresponding to the CRBs—the

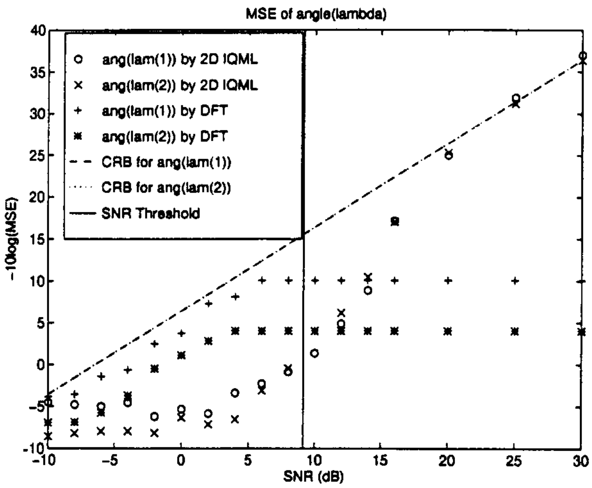

Figure 4: Threshold Behavior of Parametric and Nonparametric Techniques in Estimating $\angle(\lambda)$

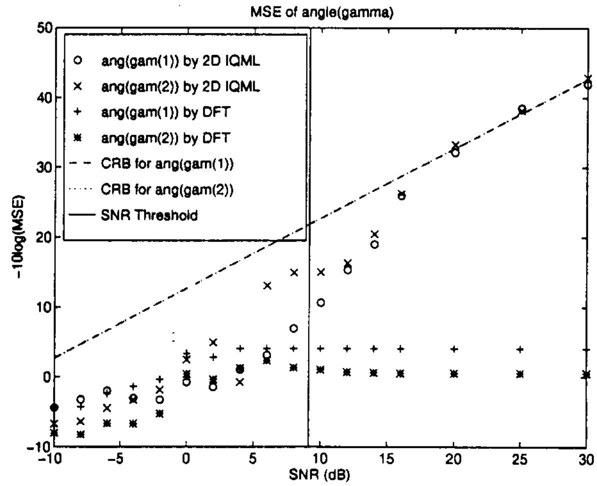

Figure 5: Threshold Behavior of Parametric and Nonparametric Techniques in estimating $\angle(\gamma)$

smallest ellipsoids associated with any unbiased estimator. Further, notice that the DFT based technique provides performance comparable (or better) than that of the parametric technique below threshold (of course the performance of the DFT method will depend on where the actual parameters lie with respect to the bins). Therefore, as the DFT is much more efficient computationally, it should be the preferred method when the SNR is below threshold.

5. CONCLUSION

In summary, algorithm independent resolution thresholds bound the SNR at which resolution is possible for an efficient estimator in a particular model. Further they allow us to answer questions concerning the SNRs at which an efficient parametric estimator achieves the Rayleigh limit. And finally, they help us to answer questions about when parametric techniques should be preferred over nonparametric techniques such as the FFT.

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