ASYMPTOTIC ANALYSIS OF AN ALGORITHM FOR IDENTIFICATION OF QUANTIZED AR TIME-SERIES

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ABSTRACT

Recently we presented a parameter estimation algorithm called the Binary Series Estimation Algorithm (BSEA) for Gaussian auto-regressive (AR) time series given 1-bit quantized noisy measurements. In this paper we carry out an asymptotic analysis of the BSEA for Gaussian AR models. In particular, from a central limit theorem we obtain expressions for the asymptotic covariances of the parameter estimates. From this we: (1) Present an algorithm for estimating the order of an AR series from one-bit quantized measurements. (2) Theoretically justify why BSEA can yield better estimates than the Yule-Walker methods in some cases.

1. INTRODUCTION AND PROBLEM FORMULATION

Recently in [4], we presented a parameter estimation algorithm called the Binary Series Estimation Algorithm (BSEA) for Gaussian auto-regressive (AR) time series given 1-bit quantized noisy measurements. Of particular interest were the surprising computer simulation results which showed that for certain AR series in multiplicative noise, the BSEA based on 1-bit quantized measurements yielded significantly better estimates than Yule-Walker methods that are based on the unquantized measurements.

In this paper we carry out an asymptotic analysis of the BSEA for Gaussian AR models. Based on the resulting asymptotic parameter variances, we address important issues such as model order estimation and theoretically explaining the surprising simulation results in [4].

Signal Model: Let $\{S_t,\ t = 0, \pm 1, \ldots\}$ be a zero mean discrete-time stationary auto-regressive (AR) process

$$S_t = \sum_{i=1}^{l} a_i S_{t-i} + Z_t, \quad Z_t \sim \text{ white } N(0, \sigma_z^2) \qquad (1)$$

where the roots of $z^l - a_1 z^{l-1} - \ldots - a_l = 0$ lie inside the unit circle. Assume that the observations X_t are obtained by sending this AR process through a randomly time-varying observation coefficient $c + W_t$ and then quantizing:

$$Y_t = (c + W_t) S_t, \quad X_t = \text{sgn}[Y_t] \qquad (2)$$

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where $W_t \sim$ white $N(0, \sigma_w^2)$, $c > 0$ and $\text{sgn}[Y_t] = 1$ if $Y_t > 0$ and 0 otherwise. Assume Z_t and W_t are mutually independent processes and σ_w is known.

Contributions: Estimating $\mathbf{a} = (a_1 \ldots a_l)'$ for a fixed order l based on $x_1, \ldots, x_n$ using BSEA was studied in [4]. Our main contributions in this paper are:

Given the n-point realization $x_1, \ldots, x_n$ of the binary series $\{X_t\}$ defined in (2):

1. Compute the asymptotic parameter estimate variance $\mathbf{E}\{(\mathbf{a} - \hat{\mathbf{a}})(\mathbf{a} - \hat{\mathbf{a}})'\}$ where $\hat{\mathbf{a}}$ denotes the BSEA parameter estimate. Because direct computation of this variance involves fourth order cumulants and so is infeasible, we use a Markovian approximation which results in a computable expression. A similar Markovian approximation is suggested in [2]. However, the Markovian approximation we use turns out to be a non-trivial generalization of that in [2].
2. Estimate the AR model order l. Based on the parameter variances we design a statistical test for selecting the model order. Our order estimation algorithm is in the same spirit as that in [3] where an order estimation of ARMA models based on unquantized observations is studied.
3. Compute the asymptotic parameter variance of the Modified Yule Walker (MYW) scheme which uses the unquantized observations $(y_1, \ldots, y_n)$ of the process $\{Y_t\}$ to estimate $\mathbf{a}$. The MYW scheme was proposed in [4] as a natural extension of the standard Yule Walker scheme to deal with the multiplicative noise. We then compare the asymptotic parameter variances of the BSEA (Objective 1) and MYW. This allows us to explain why in some cases BSEA can yield better estimates than the MYW.

Related Works: For an excellent detailed exposition of estimation algorithms for time-series based on zero crossings, see [2] and the references therein. Chapter 6,7 in [1] and Chapter 6 in [2] present a useful starting point for our analysis.

2. 1-BIT QUANTIZATION OF Y_T

Denote the correlation function of the stationary AR process $\{S_t\}$ given in (1) as

$$\rho_S(i) = \mathbf{E}\{S_t S_{t+i}\} / \mathbf{E}\{S_t^2\}, \quad \rho_S = (\rho_S(1) \ldots \rho_S(l))' \qquad (3)$$

For $i = 1, 2, \ldots$, define "transition probabilities" of X_t as

$$\lambda_i \triangleq P(X_t = 1 | X_{t-i} = 1) = P(X_t = 0 | X_{t-i} = 0) \qquad (4)$$

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Theorem 1 *Assume $i > 0$. Then the transition probabilities λ_i defined in (4) for the binary time-series $\{X_t\}$ (2) are related to the correlation $\rho_S(i)$ defined in (3) of AR process $\{S_t\}$ (1) as follows:*

$$\lambda_i = \frac{1}{2} + \frac{K_c}{\pi}\sin^{-1}\rho_S(i) \quad (5)$$

where with $\Phi(.)$ denoting the normalized Gaussian distribution, $K_c = (2F_c - 1)^2$, $F_c \triangleq P(W_t < c) = \Phi(c/\sigma_w)$.

The above theorem explicitly gives a relation between the transition probabilities λ_i of the binary series X_t and the correlation ρ_s of the AR series S_t. We now show how to compute these transition probabilities given the binary sequence realization $x_1, \ldots, x_n$.

Definition 1 *Let $D_t(i)$, $i > 0$, be the indicator function for the zero crossings spaced by i time instants of the realization $y_1, \ldots, y_n$ of the process $\{Y_t\}$: That is given a realization $x_1, \ldots, x_n$ of the binary time series $\{X_t\}$:*

$$D_t(i) = \begin{cases} 1 & \text{if } x_t \neq x_{t-i} \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

Let $D(i)$ denote the number of zero crossings spaced by i time instants: $D(i) = \sum_{t=1}^{n} D_t(i)$. Let $\mathbf{D} = (D(1)\ldots D(l))'$.

Theorem 2 *The maximum likelihood estimate of the transition probability λ_i (4), $i > 0$, based on the n-point binary time series sample path $x_1, \ldots, x_n$ of the process $\{X_t\}$ (1) is $\hat{\lambda}_i = 1 - D(i)/n$*

3. BINARY SERIES ESTIMATION ALGORITHM

Let $\hat{\rho_S}$ denote the estimated correlation based on the binary series realization $x_1, \ldots, x_n$

$$\hat{\rho_S} = (\hat{\rho}_s(1)\ldots\hat{\rho}_S(l))', \quad \hat{\rho}_S(i) = \sin\frac{\pi}{K_c}(\hat{\lambda}_i - \frac{1}{2}) \quad (7)$$

The following Binary Series Estimation (BSEA) algorithm was proposed in [4] for estimating the AR parameters $\mathbf{a}$ based on the observed binary series $x_1, \ldots, x_n$:

$$\hat{\mathbf{a}} = \hat{\mathbf{R}}^{-1}\,\hat{\rho_S} \quad (8)$$

where $\hat{\mathbf{R}}$ is the Toeplitz symmetric matrix with first row $[1 \;\; \hat{\rho}_S(1) \;\ldots\; \hat{\rho}_S(l)]$ Mean-square consistency of BSEA is proved in [4]. Let $\kappa(i,j,k)$ denote the 4-th order cumulant function of the binary series X_t and let $\gamma_X(i) = \mathbf{E}\{(X_t - 1/2)(X_{t+i} - 1/2)\}$ denote the covariance of X_t.

We now present a central limit theorem for the asymptotic normality of the zero crossings and hence the estimated covariances of the AR process $\{S_t\}$ based on the observations $\{X_t\}$. Our main contribution is to then develop a computable expression for the asymptotic variance.

Theorem 3 *The vector of zero-crossings $\mathbf{D}$ in Definition 1 satisfies the following Central Limit Theorem:*

$$n^{-1/2}(\mathbf{D} - \mathbf{E}\{\mathbf{D}\}) \Rightarrow N(0, \mu) \quad (9)$$

where $\Rightarrow$ denotes convergence in distribution and μ denotes the $l \times l$ matrix with elements

$$\mu_{ij} = \sum_{k=-\infty}^{\infty} \frac{K_c^2}{\pi^2}\left[\sin^{-1}\rho_S(k+i-j)\sin^{-1}\rho_S(k) + \sin^{-1}\rho_S(k-j)\sin^{-1}\rho_s(k+i)\right] + \kappa(i,-k,j-k) \quad (10)$$

Also the estimated correlation $\hat{\rho_S}$ computed in (7) satisfies the following Central Limit Theorem $\sqrt{n}(\hat{\rho_S} - \rho_S) \Rightarrow N(\mathbf{0}, \Sigma^x)$ where Σ^x denotes the $l \times l$ matrix with elements

$$\sigma_{ij}^x = \left(1 - \rho_S^2(i)\right)^{1/2}\left(1 - \rho_S^2(j)\right)^{1/2}\frac{\pi^2}{K_c^2}\mu_{ij} \quad (11)$$

Theorem 4 *The parameter estimate $\hat{\mathbf{a}}$ computed in (8) based on the observations $x_1, \ldots, x_n$ satisfies*

$$\sqrt{n}(\hat{\mathbf{a}} - \mathbf{a}) \Rightarrow N(0, \Xi^x) \;\; \text{where } \Xi^x \triangleq \frac{\partial \mathbf{a}}{\partial \rho_S}\Sigma^x\frac{\partial \mathbf{a}}{\partial \rho_S}' \quad (12)$$

Ξ^x is a $l \times l$ matrix with elements ξ_{ij}^x.

Remark: Notice that Ξ^x in Theorem 4 is messy to compute since $\partial\mathbf{a}/\partial\rho_S$ needs to be evaluated analytically (this can be done using symbolic packages like MAPLE). For this reason, when presenting an order estimation algorithm in Sec.5, we shall work with the asymptotic variance of the correlations Σ^x of Theorem 3 [1] Note that for the AR(1) case $\partial\mathbf{a}/\partial\rho_S = 1$.

4. MARKOVIAN VARIANCE APPROXIMATION

In general it is extremely difficult to compute the zero-crossings variance μ_{ij} (10) due to the presence of the 4-th order cumulant term κ. However, as shown in [2], is many cases it is reasonable to approximate $D_t(1)$ by a first-order 2 state Markov chain. In particular, for continuous spectrum processes (such as ours), this Markovian approximation holds surprisingly well (see [2], pg 180 for details). In this section we use this Markovian assumption to obtain a computable expression for μ and hence to compute Σ^x

Assumption 1 *$D_t(1) \in \{0,1\}$ defined in (6) is a discrete-time two-state first-order homogeneous Markov chain with transition probability matrix $\mathbf{V} = (V_{ij})$, $i,j \in \{0,1\}$ and $V_{11} = v_0 = P(D_t(1) = 0 | D_{t-1}(1) = 0)$, $V_{22} = v_1 = P(D_t(1) = 1 | D_{t-1}(1) = 1)$ and steady state probabilities $\pi = (\pi_0 \;\; \pi_1)' = (q \;\; p)'$ where*

$$(q \;\; p)' \triangleq (P(D_t(1) = 0) \;\; P(D_t(1) = 1))' \quad (13)$$

Assumption 1 yields closed form expressions for μ_{ij} and hence Σ^x. Also the asymptotic variances are positive by construction.

Lemma 1 *The transition probabilities $\mathbf{V}$ and steady state probabilities π (13) of the Markov chain $D_t(i)$ can be expressed in terms of the transition probabilities λ_i (4) of the binary time-series X_t as $\pi = (\lambda_1 \;\; 1 - \lambda_1)'$, $v_0 = (2\lambda_1 + \lambda_2 - 1)/(2\lambda_1)$, $v_1 = (1 - 2\lambda_1 + \lambda_2)/(2(1 - \lambda_1))$.*

[1] Σ^x is the asymptotic variance per unit data length.

Characterization of offspring sequences: In order to compute μ_{ij}, it is necessary to first characterize the event $\{D_t(i)=1\}$ in terms of sequences of $D_.(1)$.

Definition 2 *If a particular sequence of* $(D_t(1),\ldots,D_{t-i}(1))$ *is equivalent to the event* $\{D_t(i)=1\}$, *we call such a sequence an "offspring" of the event* $\{D_t(i)=1\}$. *Let* C_i, $i\geq 1$, *denote the set of these* 2^{i-1} *possible offspring, i.e., if* $g=(g_1,\ldots,g_i)$ *is an i-length binary sequence then*

$$g\in C_i \;\; \text{if} \;\; (D_t(1)=g_1,\ldots,D_{t-i}(1)=g_i) \;\; \text{implies} \;\; D_t(i)=1 \tag{14}$$

We have the following characterization of C_i.

Lemma 2 *An i-length binary sequence* $g=(g_1,\ldots,g_i)\in C_i$ *if*

$$(g_1+(i-1)[g_2+g_3+\cdots+g_{i-1}]+g_i) \bmod 2 = 1 \tag{15}$$

Clearly C_i has 2^{i-1} elements because out of the 2^i possible i-length binary sequences, only 2^{i-1} of them which satisfy Lemma 2 belong to C_i.

Computation of μ: The following is a constructive method for computing $\mu_{ij}=\text{cov}\{D(i),D(j)\}$ defined in (9), (10): Denote $\gamma_D^{ij}(t-k)\triangleq\text{cov}\{D_t(i),D_k(j)\}$.

Theorem 5 *Under Assumption 1, for sufficiently large n,* μ_{ij}, $;i,j\in\{1,2,\ldots,\}$ *in (9) is computed as*

$$\mu_{ij}=\gamma_D^{ij}(0)+\sum_{t=1}^{j-2}\gamma_D^{ij}(t)+\sum_{t=1}^{i-2}\gamma_D^{ij}(-t)+\frac{\eta_{ij}^{+}+\eta_{ij}^{-}}{1-\alpha} \tag{16}$$

where $\alpha=(v_1-p)/q$, *and*

$$\eta_{ij}^{+}=\sum_{g\in C_i}\sum_{h\in C_j}V_{h_2h_1}V_{h_3h_2}\cdots V_{h_jh_{j-1}}(2\,\delta(h_j-g_1)-1)$$
$$(1-\pi_{g_1})V_{g_2g_1}V_{g_3g_2}\cdots V_{g_ig_{i-1}}\pi_{g_i}$$
$$\eta_{ij}^{-}=\sum_{g\in C_i}\sum_{h\in C_j}V_{h_2h_1}V_{h_3h_2}\cdots V_{h_jh_{j-1}}(2\,\delta(g_j-h_1)-1)$$
$$(1-\pi_{h_1})V_{g_2g_1}V_{g_3g_2}\cdots V_{g_ig_{i-1}}\pi_{h_j} \tag{17}$$

In (17), $\delta(x)=1$ *if* $x=0$ *and* 0 *otherwise.*

Remarks:
1. When $i=j$, then $\gamma_D^{ii}(t)=\gamma_D^{ii}(-t)$ and $\eta_{ii}^{+}=\eta_{ii}^{-}$. So denoting $\eta_{ii}=\eta_{ii}^{+}=\eta_{ii}^{-}$ we have from (16)

$$\mu_{ii}=\gamma_D^{ii}(0)+2\sum_{t=1}^{i-2}\gamma_D^{ii}(t)+\frac{2\,\eta^{ii}}{1-\alpha},\quad i\in\{1,2,\ldots,\} \tag{18}$$

2. It is straightforward to compute η_{ij}^{+} and η_{ij}^{-} in (17) numerically. For example when $i,j=1$ we have

$$\mu_{11}=\text{var}\{D(1)\} \;=\; p\,q\left(\frac{1-2p+v_1}{1-v_1}\right) \tag{19}$$

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We now use Theorem 5 (along with Theorems 3, 4) to estimate the order of AR processes given one one-bit quantized data $\{X_t\}$ (Sec. 5), and to explain why BSEA can yield better estimates than MYW in some cases (Sec. 6).

5. ORDER ESTIMATION USING $\{X_T\}$

Given observations $(x_1,\ldots,x_n)$ of the binary time-series $\{X_t\}$, based on Theorems 3 and 5 the following algorithm to test for an AR(l) model within the class of AR models:

1. Compute the asymptotic variance of correlation $\Sigma^x=(\sigma_{ij}^x)$ defined in (11). This consists of:

(i) For $i=1,\ldots,l$, estimate $\hat{\lambda}_i$ and hence $\hat{\rho}_S(i)$, $\hat{a}_i$ using BSEA of Sec.3.

Then assume $\lambda_i=\hat{\lambda}_i$, $a_i=\hat{a}_i$ and $\rho_S(i)=\hat{\rho}_S(i)$ for $i=1,\ldots,l$. (see Remark 2 below).

(ii) For $i>l$, compute $\rho_S(i)=\sum_{j=1}^{l}a_j\,\rho_S(i-j)$, λ_i using (5) and $\sigma_{ii}^x=(1-\rho_S^2(i))\pi^2\,\mu_{ii}/K_c^2$ where μ_{ii} is computed using Theorem 5, (18).

The terms $\gamma_D^{ii}(t)$, $t=0,\ldots,i-2$ in (18) are computed as

$$\begin{aligned}\gamma_D^{ii}(0) &= \lambda_i(1-\lambda_i)\\ \gamma_D^{ii}(t) &= \sum_{g\in C_i}\sum_{h\in C_i}v_{h_2h_1}\cdots v_{h_{t+1}h_t}[\delta(h_{t+1}-g_1)\\ &\quad \delta(h_{t+2}-g_2)\ldots\delta(h_i-g_{i-t})-v_{h_{t+2}h_{t+1}}\cdots\\ &\quad v_{h_ih_{i-1}}\pi_{h_i}v_{g_2g_1}\cdots v_{g_{i-t}g_{i-t-1}}]\\ &\quad v_{g_{i-t+1}g_{i-t}}\cdots v_{g_ig_{i-1}}\pi_{g_i}\end{aligned} \tag{20}$$

2. Compute the sample correlations $\hat{\rho}_S(i)$, $i>l$ based on $(x_1,\ldots,x_n)$ using BSEA.

If $\hat{\rho}_S(i)$, $i>l$ lie between the limits $\rho_S(i)\pm1.96\,(\sigma_{ii}^x/n)^{1/2}$ then from Theorem 3 we can infer with 95% confidence that the series is generated by an AR(l) model.

Typically, given the binary data $(x_1,\ldots,x_n)$, start the test with a AR(0) model and increment the model order until Step 2 above holds. The minimum model order $\hat{l}$ for which Step 2 first holds is taken to be the model order (within the class of AR models).

Remarks:
1. *White noise test*: Actually for an AR(0) model the above test is somewhat simplified as follows: For an AR(0) model, i.e., $S_t=Z_t$, $p=q=v_1=v_2=0.5$ and so $\mu_{ii}=1/4$ and $\mu_{ij}=0$, $i\neq j$. So based on Theorem 5, if $\hat{\rho}_S(i)$, $i=1,2,\ldots,$ lie between $\pm1.96\,n^{-1/2}\,\pi/(2\,K_c)$ then we can say with 95% confidence that the series is white noise.
2. The assumption in Step 1 i) that $a_i=\hat{a}_i$, etc for $i\leq l$ is standard in preliminary order estimation, see[3], Chapters 7,8,9. Note that all the variables in Step 1 are computed in terms of these "true" parameters a_i, λ_i, ρ_i, $i\leq l$.

6. ASYMPTOTIC VARIANCE OF MYW AND COMPARISON WITH BSEA

The MYW algorithm yields estimates of **a** based on the un-quantized observations $y_1,\ldots,y_n$ of the process $\{Y_t\}$ defined in (2).

A consistent estimate of $\rho_S(i)$ based on $y_1,\ldots,y_n$ is:

$$\tilde{\rho}_S(i)=\left(1+\sigma_w^2/c^2\right)\left(\sum_{t=1}^{T}y_t\,y_{t-i}/\sum_{t=1}^{T}y_t^2\right),\quad i\neq0 \tag{21}$$

The MYW uses the Yule-Walker equations (8) with $\tilde{\rho}_S(i)$ instead of $\hat{\rho}_S(i)$ where $\tilde{\rho}_S(i)$ is evaluated in (21).

Let $\tilde{\mathbf{a}}$ denote the MYW estimate of **a** based on $y_1,\ldots,y_n$.

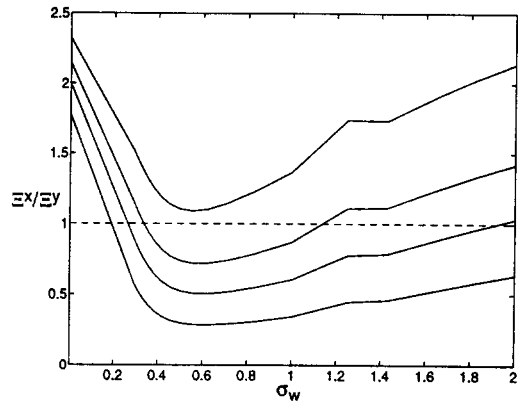
Theorem 6 *For an AR(1) model, the MYW estimate* $\tilde{a} = \tilde{a}_1$ *is distributed normally as* $\sqrt{n}(\tilde{a} - a) \Rightarrow N(0, \Xi^y)$ *where*

$$\Xi^y = \frac{a^2}{(1-a^2)}\left(a^2 + 1 - \frac{2a^2}{(1+\sigma_w^2)}\right)^2 + \left(a^2 + 1 - 2\frac{a^2}{(1+\sigma_w^2)} + \sigma_w^2\right)^2 \quad (22)$$

Comparison of BSEA with MYW: Based on the expressions (19) and (22), we can draw the following conclusions:

1. Unlike MYW where when $\sigma_w > 0$, $\Xi^y \to \infty$ as $a \to 1$, for the BSEA $\Xi^x \to 0$ as $a \to 1$. So as $a \to 1$, $\Xi^y > \Xi^x$ meaning that BSEA yields more accurate estimates than MYW for a fixed data length.
2. When $a \to 0$ and $\sigma_w = 0$ then from (22) and (12), (19) we have $\Xi^x/\Xi^y = \pi^2/4$ which is the result given in [1]. That is, in zero multiplicative noise and for small a, BSEA required roughly twice ($= \pi^2/4$) the data length to yield comparable estimates to MYW.
3. In Figure 1 we plot the Ξ^x/Ξ^y versus σ_w for various values of a. When the ratio is less than one it means that BSEA performs better than MYW.

For example when $a = 0.9$ and $\sigma_w = 1$, $\Xi^x/\Xi^y \approx 0.5$. This means that MYW would require twice the data length to achieve the accuracy of BSEA. For $a = 0.85, 0.8$ the range σ_w over which BSEA does better than MYW gets smaller. When $a = 0.7$ MYW always does better than BSEA, although for $\sigma_w \approx 0.6$ the estimates are comparable with BSEA.



$a = 0.9$ (lowest curve), 0.85, 0.8, 0.7 (highest curve)

Figure 1: Theoretical ratio of asymptotic BSEA variance Ξ^x to MYW variance Ξ^y for AR(1) Model

7. SIMULATION EXAMPLE

Simulation results on the performance of BSEA for parameter estimation are presented in [4]. Our aim here is to present an example of the order estimation algorithm.

AR(4) example: $\mathbf{a} = (0.4 \ \ -0.4 \ \ -0.4 \ \ 0.2)'$, $\sigma_v = 1.0$, $\sigma_w = 0.0$, $n = 20{,}000$. Table 1 shows the tests for AR(2), AR(3) and AR(4) models, respectively. The estimated correlations lie within the 95% confidence upper and lower bounds only for the AR(4) test; showing that within the class of AR models, we can infer with 95 % confidence that the underlying model is AR(4). Indeed, for the AR(2) and AR(3) case, none of the estimated correlations lie within the bounds.

Note: For detialed proofs and simulations see [5].

AR(2) Model test

i	3	4	5	6	7
upper	-0.5028	0.0801	0.4133	0.2038	-0.1493
$\hat{\rho}s(i)$	-0.6857	-0.0099	0.5422	0.3934	-0.2039
lower	-0.5359	0.0365	0.3755	0.1618	-0.1924

AR(3) Model test

i	4	5	6	7	8
upper	-0.0663	0.5103	0.4657	-0.0474	-0.4002
$\hat{\rho}s(i)$	-0.0099	0.5422	0.3934	-0.2039	-0.4510
lower	-0.1098	0.4745	0.4275	-0.0910	-0.4385

AR(4) Model test

i	5	6	7	8	9
upper	0.5642	0.4257	-0.1680	-0.4486	-0.1487
$\hat{\rho}s(i)$	0.5422	0.3934	-0.2039	-0.4510	-0.1557
lower	0.5298	0.3867	-0.2110	-0.4859	-0.1920

Table 1: Order Estimation for AR(4) Model, $a = (0.4 \ -0.4 \ -0.4 \ 0.2)'$

8. REFERENCES

[1] B. Kedem, *Binary Time Series*, Lecture Notes in Pure and Applied Mathematics, Vol.52, Marcel Dekker, 1980.

[2] B. Kedem, *Time Series Analysis by Higher Order Crossings*, IEEE Press, 1994.

[3] P.J. Brockwell and R.A. Davis, *Time Series: Theory and Methods*, Second Edition, Springer Series in Statistics, Springer Verlag, 1991.

[4] V.Krishnamurthy and I.M.Y. Mareels, *Estimation of noisy 1-bit Quantized AR Time-series with randomly varying observation coefficient*, IEEE Trans Signal Processing, 1994, to appear May 1994.

[5] V. Krishnamurthy and H.V. Poor, *Asymptotic Analysis of an Algorithm for Parameter Estimation and Identification of 1-bit Quantized AR Time-Series*, submitted to IEEE Trans Signal Proc., 1994.