STATE DECOUPLING IN ESTIMATION THEORY

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Abstract

When a system is unobservable, the error covariance associated with a Kalman filter will be nearly singular. As a consequence, an optimum estimation does not exist.

In this paper, we show that this system can be transformed into a nonlinear system with a linear measurement equation. In addition to other useful features, this transformation also serves to decouple the state in such a way that an observable part can be extracted and estimated while no information can be gained and processed for the unobservable part.

1 Introduction

The concept of observability was originally introduced by Kalman for linear deterministic systems [1]. A system is defined to be observable if the state of the system can be uniquely determined from past observations. A linear system is observable if and only if the rank of the corresponding observability matrix is equal to the dimension of the state vector. The concept of observability has also been extended to the nonlinear systems. For nonlinear case, most of the results obtained are true in local sense [2].

In Kalman filtering theory, the observability of a linear stochastic system implies that the error covariance matrix is stably bounded and converges to a steady state [3]. The Kalman error covariance satisfies a matrix Riccati equation whose stability can be demonstrated from a stochastic observability matrix, *i.e.* the Fisher information matrix (FIM).

When a system is unobservable, the error covariance associated with a Kalman filter will be nearly singular, exhibiting some unstable behavior or become "ill-conditioned." Unstable behavior implies that the error covariance matrix will be nearly singular, leaving some exceedingly large elements. In such a case, the given observation or measurement equation does not provide a set of sufficient statistics for the state. As a consequence, an optimum estimation does not exist.

For a nonlinear system, it is difficult to give a precise definition of observability. Typically, an extended Kalman filter (EFK) is employed to generate simulation results. Observability of the nonlinear system is assumed to exist if and only if the estimation and the error covariance converge. In this paper, we deal with a linear deterministic system with nonlinear measurements corrupted by additive white Gaussian noise. A property of this system is that it can be transformed into a nonlinear system with a linear measurement equation. The original system is unobservable. However, the new state consists distinctively of two parts: an observable part and an unobservable part. In other words, this transformation serves to decouple the state in such a way that an observable part can be extracted and estimated while no information can be gained and processed for the unobservable part.

In the next section, a mathematical description of the dynamic system is given. Observability is defined via FIM. In Section 3, the technique of a nonlinear transformation that leads to the decoupling of state estimation is explained. Finally an application of the technique to a three dimensional target tracking problem with conical angle measurements is illustrated in Section 4.

2 System Description

We consider a linear system described by

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k), \qquad k = 0, 1, \cdots \quad (1)$$

where $\mathbf{x} \in R^n$ is a $n \times 1$ state vector and $\mathbf{\Phi}$ is an $n \times n$ state transition matrix [1]. The initial state vector $\mathbf{x}(0)$ is a normal random vector with zero mean and covariance $\mathbf{P}_0$.

The observation ($\mathbf{y}(k)$ on $\mathbf{x}(k)$) is made via the nonlinear equation

$$\mathbf{y}(k) = \mathbf{h}[\mathbf{x}(k)] + \mathbf{w}(k) \tag{2}$$

where $\mathbf{h} : R^n \to R^m$ is assumed to be continuously differentiable, $\mathbf{w}(k)$ is a zero-mean white Gaussian sequence with a covariance given by

$$E[\mathbf{w}(k)\mathbf{w}(j)'] = \mathbf{R}\delta(k, j)$$

where the prime denotes the transpose of a vector, $\delta(k, j)$ is the Kronecker delta and $\mathbf{R}$ is an $m \times m$ positive-definite symmetric matrix.

For the observation system described by (1) and (2), the FIM is defined as [3]

$$\mathbf{J}_1(k) = \sum_{i=1}^{k} \left[(\mathbf{\Phi}^{i-k})' (\frac{\partial \mathbf{h}'}{\partial \mathbf{x}} \mathbf{R}^{-1} \frac{\partial \mathbf{h}}{\partial \mathbf{x}}) (\mathbf{\Phi}^{i-k}) \right] \tag{3}$$

evaluated along the trajectory $\mathbf{x}(i)$ for $i = 1, 2, \cdots$. $\mathbf{J}_1$ is a nonnegative definite symmetric matrix. In recursive estimation theory, $\mathbf{J}_1$ provides a measurement of the observability of the system. The significance of the FIM can be seen, for example from its relationship to the Cramer-Rao Lower Bound (CRLB) $\mathbf{P}_x^*$. The latter has the property, for any unbiased estimate $\hat{\mathbf{x}}$, the error covariance $\mathbf{P}_x$ is no less than $\mathbf{P}_x^*$, *i.e.*

$$\mathbf{P}_x \geq \mathbf{P}_x^*.$$

The CRLB therefore represents the "best" that can be achieved by an unbiased estimator along a particular trajectory. The FIM and CRLB are related through the following [4].

$$\mathbf{P}_x^{*-1}(k) = (\mathbf{\Phi}^{-k})' \mathbf{P}_0^{-1} (\mathbf{\Phi}^{-k}) + \mathbf{J}_1(k). \tag{4}$$

Without loss of generality, we let $\mathbf{P}_x^{*-1}(0) = 0$ (see [3] p.231) and obtain

$$\mathbf{P}_x^{*-1}(k) = \mathbf{J}_1(k). \tag{5}$$

A large $\mathbf{J}_1$ implies a small $\mathbf{P}_x^*$ and hence a small error covariance [2]in actual simulation. In this sense, the FIM is a measurement of observability. A system is said to be *completely observable* if and only if $\mathbf{P}_x^{*-1}(k) = \mathbf{J}_1(k)$ is *positive definite* for some $k > 0$ [3].

We now define a system to be unobservable if $|\mathbf{J}_1(k)| = 0$ for all k and along $\mathbf{x}(\cdot)$. From (5), we now have

$$\mathbf{P}_x^{*-1}(k) = 0$$

and $\mathbf{P}_x^*$ is ill-conditioned, containing some extremely large elements.

It has been shown [4] that the CRLB satisfies equations identical to those satisfied by the error covariance of EKF. Consequently, when $|\mathbf{J}_1| = 0$, the error covariance becomes unstable and the filtering (or estimation) algorithm degenerates. Thus, an unobservable system is manifested by the unwieldy collapsing of the error covariance of EKF.

3 State Decoupling

Suppose there is a function $\mathbf{g} : R^n \to R^n$ such that

(i) the inverse function $\mathbf{g}^{-1}$ exists, and

(ii)

$$\mathbf{h}[\mathbf{g}^{-1}(\mathbf{z})] = \mathbf{H}\mathbf{z} \tag{6}$$

where $\mathbf{H}$ is a constant matrix.

Let

$$\mathbf{z} = \mathbf{g}(\mathbf{x}) \tag{7}$$

and hence

$$\mathbf{x} = \mathbf{g}^{-1}(\mathbf{z}). \tag{8}$$

Substituting (8) into the state equation (1) yields

$$\mathbf{g}^{-1}[\mathbf{z}(k+1)] = \mathbf{\Phi}\mathbf{g}^{-1}[\mathbf{z}(k)] \tag{9}$$

from (7) and (9), we have a new state equation

$$\mathbf{z}(k+1) = \mathbf{f}[\mathbf{z}(k)] \tag{10}$$

where

$$\mathbf{f} = \mathbf{g}[\mathbf{\Phi}\mathbf{g}^{-1}]. \tag{11}$$

The measurement equation (2) now becomes

$$\mathbf{y}(k) = \mathbf{H}\mathbf{z}(k) + \mathbf{w}(k) \tag{12}$$

The original model of a linear system (1) with nonlinear measurements (2) are now transformed into a second model with nonlinear system (10) and linear measurements (12). this type of transformation was

[1] $\mathbf{\Phi}$ is time-dependent, *i.e.* $\mathbf{\Phi} = \mathbf{\Phi}(k+1, k)$. But for brevity, we do not indicate this dependence.

[2] Here, the magnitude of a matrix is measured by its determinant.

employed in the development of the modified spherical EKF [3]. Let us now study this transformation on the observability.

Let $\mathbf{J}_2(k)$ be the FIM of the new (transformed) system (10) and (12), *i.e.*

$$\mathbf{J}_2(k) =$$

$$\sum_{i=1}^{k}\left[(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{\Phi}\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1})^{k-i}\right]'(\mathbf{H}'\mathbf{R}^{-1}\mathbf{H})\left[(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{\Phi}\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1})^{k-i}\right] \tag{13}$$

It then follows from (3) and (13) that $\mathbf{J}_1$ and $\mathbf{J}_2$ are related as follows

$$\mathbf{J}_1 = \frac{\partial \mathbf{g}'}{\partial \mathbf{x}}\mathbf{J}_2\frac{\partial \mathbf{g}}{\partial \mathbf{x}}. \tag{14}$$

When the original system is unobservable, *i.e.* $|\mathbf{J}_1| = 0$, then clearly, the transformed system is still unobservable ($|\mathbf{J}_2| = 0$).

However, under some conditions, the state $\mathbf{z}$ can be decoupled into two components: an observable component $\mathbf{z}_1$ of dimension p and an unobservable component $\mathbf{z}_2$ of dimension $n-p$, where p is the rank of $\mathbf{J}_2$. This is the main result of our paper which will be shown in the following.

First, due to the fact that $\mathbf{J}_2$ is a non-negative definite symmetric matrix, we can express $\mathbf{J}_2$ as [3]

$$\mathbf{J}_2 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{15}$$

where $\mathbf{I}_2$ is a $p \times p$ positive definite matrix.

We then assume that

$$\mathbf{H} = [\mathbf{H}_1 \ \mathbf{0}] \tag{16}$$

and

$$\left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\mathbf{\Phi}\frac{\partial \mathbf{g}}{\partial \mathbf{z}}^{-1}\right) \doteq \mathbf{F} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_2 \\ \mathbf{0} & \mathbf{F}_3 \end{bmatrix} \tag{17}$$

where $\mathbf{H}_1$ is $m \times p$, $\mathbf{F}_1$ is $p \times p$ and $\mathbf{F}_2$ and $\mathbf{F}_3$ are matrices of appropriate dimensions.

Substituting (16) into (12), we have

$$\mathbf{y}(k) = \mathbf{H}_1\mathbf{z}_1(k) + \mathbf{w}(k). \tag{18}$$

Also, (16), (17), and (18) suggest that Kalman filter equations can be written for $\mathbf{z}_1$, independent of $\mathbf{z}_2$. Especially, the discrete matrix Riccati equation for $\mathbf{P}^*_{\mathbf{z}_1}(k)$ now satisfies the equation

$$\mathbf{P}^*_{\mathbf{z}_1}(k+1) = \mathbf{F}_1\mathbf{P}^*_{\mathbf{z}_1}(k)\mathbf{F}_1' - \mathbf{F}_1\mathbf{P}^*_{\mathbf{z}_1}(k)\mathbf{H}_1'\times$$

$$[\mathbf{H}_1\mathbf{P}^*_{\mathbf{z}_1}(k)\mathbf{H}_1' + \mathbf{R}]^{-1}\mathbf{H}_1\mathbf{P}^*_{\mathbf{z}_1}(k)\mathbf{F}_1'. \tag{19}$$

As stated in [4], the EKF covariance propagation equations *linearized about the true unknown trajectory* provide the CRLB. Therefore, $\mathbf{P}^*_{\mathbf{z}_1}$ in (19) is the CRLB of $\hat{\mathbf{z}}_1$.

In (19), $\mathbf{H}_1'\mathbf{R}^{-1}\mathbf{H}_1$ is positive definite. It can then be shown that its solution $\mathbf{P}^*_{\mathbf{z}_1}$ is bounded, *i. e.*

$$|\mathbf{P}^*_{\mathbf{z}_1}(k)| \leq \delta \tag{20}$$

regardless of the trajectory $\{\mathbf{x}(k)\}_{k=0}^{\infty}$ along which $\mathbf{F}_1$ is evaluated. Therefore, for any k

$$\mathbf{P}^{*\ -1}_{\mathbf{z}_1}(k) \approx \mathbf{I}_2 > 0 \tag{21}$$

which shows that $\mathbf{P}^*_{\mathbf{z}_1}(k)$ is not ill-conditioned and the EKF filtering algorithm for $\mathbf{z}_1$ will not degenerate.

It can be seen that the key role in decoupling is played by the function $\mathbf{g}$. The proper choice for $\mathbf{g}$ is motivated by some physical consideration involving transformation of coordination systems. One of the earlier papers that dealt with this subject is by Holeezer, Jolsson, and Cohen [6] in which, their discussed the modified polar coordinates in bearings-only target ranging. A more recent report appeared in [7] and is briefly mentioned in the next section as an example.

Even though observability on $\mathbf{z}_1$ does not necessarily imply observability on any components of the original state $\mathbf{x}$, it can be argued that such a decoupling enables us to gain observability on a subspace of the transformed space $\mathbf{z}$. This will often provide useful information and insight into the original estimation problem as can be seen in the tracking problem mentioned above.

4 Applications to the Tracking Problem

Considering an underwater tracking system problem [5], let $(R_x, R_y, R_z)'$ be the relative position (in Cartesian coordinates) between ownship and target, and let $(V_x, V_y)'$ be the relative horizontal velocity. We assume ownship are target both travel at constant velocity and depth ($V_z = 0$). the state vector is $\mathbf{x} = (R_x, R_y, R_z, V_x, V_y)'$. Then $\mathbf{\Phi}$ satisfies the

[3]Some elementary row operations, which may be needed to bring $\mathbf{J}_2$ into this form are not included in the discussion here.

state equation (1) when Φ is given by

$$\Phi = \begin{bmatrix} 1 & 0 & 0 & T & 0 \\ 0 & 1 & 0 & 0 & T \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$

and T is the sample time interval. Ownship's observation of $\mathbf{x}$ is given by the conical angle measurements

$$y(k) = \theta(k) + w(k) \tag{23}$$

where (see Fig. 1)

$$\theta = \cos^{-1}\left\{\frac{R_x}{(R_x^2 + R_y^2 + R_z^2)^{\frac{1}{2}}}\right\}. \tag{24}$$

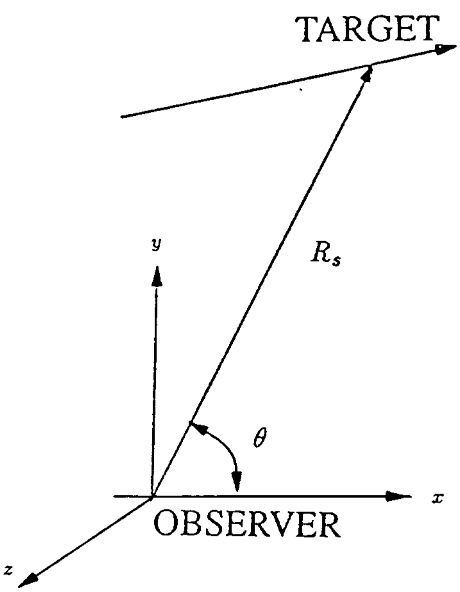

Figure 1: Root Mean-Squared Error vs. SNR

We now have a linear system with nonlinear measurements from which $\mathbf{x}$ is to be estimated.

The FIM $\mathbf{J}_1$ for this system can be computed from (3), (22), and (24) by first evaluating the partial derivatives of θ with respect to R_x, R_y, and R_z. It can be shown that $|\mathbf{J}_1(k)| = 0$, *i.e.* the system is not observable.

The modified spherical coordinate system has been seen for a tracking system of this type. In this system, the transformed state is given by

$$\mathbf{z} = (\theta,\ \dot{\theta},\ \dot{\rho},\ \frac{1}{R_s},\ \phi)' \tag{25}$$

where $\dot{\theta}$ is the conical angle rate, $\dot{\rho} = \frac{\dot{R}_s}{R_s}$, $R_s = (R_x^2+R_y^2+R_z^2)^{\frac{1}{2}}$, and $\phi = \tan^{-1}(R_y/R_x)$. Expression for g can be found in [7].

In terms of $\mathbf{z}$, the measurement equation can be written as

$$y(k) = \mathbf{H}\mathbf{z}(k) + n(k) \tag{26}$$

where $\mathbf{H} = (1,\ 0,\ 0,\ 0, 0)$. It can then be shown that conditions for decoupling (16) and (17) are with $\mathbf{F}_1$ being a 3×3 matrix. We therefore conclude that the first three components of the modified spherical coordinate vector $\mathbf{z}$ are observable. Even though the entire state, either $\mathbf{x}$ and $\mathbf{z}$, is not observable. The same conclusion has been reached before [8] but our approach, which is based on the FIM and decoupling of the state provides a more solid theoretical foundation and can be applied to a broad class of models.

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